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CONTENTS

Topological rigidity theorems for conformal transformation groups of C,0 and the diagrams of analytic correspondences

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Abstract. We classify solvable groups of diffeomorphisms of C,0, and prove topological rigidity theorems for the various groups of diffeomorphisms and apply to the diagrams of analytic correspondences (germs of algebraic functions). We define also the separatrix for group actions and discuss the structure of orbits of non-solvable groups and correspondences.

Introduction. The topology of conformal transformation groups of the germ of $\mathbb{C}$ at the origin is regarded from many different points of view, cf. the monodromy groups of differential equations, the projective holonomy of singular 1-forms [5,11] and the non isolated singularities of map germs [14]. Recently it is pointed out that the actions of solvable groups possess special topological properties by Il'yashenko and Shcherbakov [8,16]. In this paper we first classify these actions. Using it we prove the topological rigidity of parametrized families of groups, and apply to that of the diagrams of correspondences (multi-valued functions).

The rigidity was first regarded by Il'yashenko [8]. He considered the monodromy groups of algebraic differential equations of the complex projective plane along the line at infinity which is a special compact solution distinguished from the others. He then observed that under a certain condition there exist at least n-1 distinguished topological invariants for the groups generated by n germs of diffeomorphisms of $\mathbb{C},0$, topological conjugacy implies analytic conjugacy, and he interpreted these results in terms of algebraic differential equations. However many details of the theory were left open. Later some of those problems were proved by Shcherbakov [16] for non solvable groups.

A similar but completely independent study was carried on by Cerveau and Sad [5] from the view point of the 1-forms of $\mathbb{C}^2$. Namely they considered the blow up $\tilde{\omega}$ on $\tilde{\mathbb{C}}^2$ (reduced) of a 1-form ω on $\mathbb{C}^2,0$. The *projective holonomy* of ω is the holonomy of $\tilde{\omega}$ along P^1-sing $\tilde{\omega}$ (here we assume the exceptional curve P^1 is a leaf). They proved that the projective holonomy is a topological invariant under a certain condition. Recently it is shown that most finitely generated groups acting on $\mathbb{C},0$ are realized as the projective monodromy by Lins Neto [11].

For open (flat) morphisms f of $\mathbb{C}^3,0$ to $\mathbb{C}^2,0$ of a certain special form singular along the fibers $f^{-1}(0)$, the author [14] defined commutative groups G(f) acting on the fibers, and proved that the group is invariant under topological conjugacy and presented some invariants explicitly. This method will be generalized further in a coming paper.

These individual studies are based on the topology of groups of diffeomorphisms of $\mathbb{C},0$. In this paper we consider the various topological rigidity theorems for the groups and the diagrams of correspondences. First we prove a rigidity theorem for groups generated by holomorphic families of diffeomorphisms of $\mathbb{C},0$ (Theorem 4.1).

A *correspondence of* $\mathbb{C},0$ *to* $\mathbb{C},0$ is a germ of an analytic curve $\Gamma \subset \mathbb{C} \times \mathbb{C}$. This notion generalizes the germs of algebraic functions. The group $G(\Gamma)$ is generated by the monodromy actions of the compositions of the normalization $\tilde{\Gamma} \to \Gamma$ with the projections onto the first and the second $\mathbb{C}$. We apply the rigidity theorem to the $G(\Gamma)$ acting on $\tilde{\Gamma}$ and prove a rigidity theorem for the diagrams of families of correspondences (Theorems 5.1,5.3).

The geometry of diffeomorphisms of $\mathbb{C},0$ has been long studied. However the complete account of fundamental results is not seen in a systematic text. In Section 1 we revise those results including the method due to Fatou, Kimura and Ecalle, etc quickly in a form involving the residue invariant m(f). The residue invariant m(f) for diffeomorphism f of $\mathbb{C},0$ seems to be firstly defined geometrically in this paper. The residue presents a geometric interpretation of formal equivalence. We enjoy some multiplicative formulae for the residue. In Section 2 we classify commutative as well as solvable groups (Theorems 2.1, 2.2) involving the residue. In Section 4 we prove the rigidity theorem for the groups (Theorem 4.1), and in Section 5 we apply the rigidity to that of the diagrams of analytic correspondences of $\mathbb{C},0$ (Theorems 5.1,5.3). Here the explicit classification of the groups in Section 2 is used. These results are all to be interpreted in a coming paper to classify flat morphisms of $\mathbb{C}^n,0$ to $\mathbb{C}^2,0$ with the singular locus $f^{-1}(0) \cap \Sigma(f)$ of dimension one. Finally in Section 6 we discuss some problems on the orbit structure of group actions as well as global correspondences of Riemannian surfaces.

The author is informed that Cerveau and Moussu [4] obtained analogous results for slovable groups.

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1. Preliminaries

We begin by introducing the method due to Fatou, Kimura, Ecalle, Voronin and Malgrange etc [6,7,9,17]. On the k-sheet covering $\mathbb{C}_k$ of $\mathbb{C} - 0$ a k-flat diffeomorphism $f(z) = z + a_{k+1}z^{k+1} + \ldots$ lifts to a diffeomorphism F, which is presented with the coordinate $\tilde{z} = z^{-k}$

$$F(\tilde{z}) = \tilde{z} - a_{k+1}\ k + a'\tilde{z}^{-1/k} + \ldots \qquad .$$

from which we have

$$|F(\tilde{z}) - (\tilde{z} - a_{k+1}\ k)| \le c|\tilde{z}|^{-1/k}$$

for $|\tilde{z}|$ sufficiently large with a positive constant c. This shows that if arg $\tilde{z}$ is close to $\arg(-a_{k+1}\ k)$ and $|\tilde{z}|$ is sufficiently large, the orbit of $\tilde{z}$ under F is contained in an arbitrary thin cone

$$\{\tilde{z} + \omega \in \mathbb{C}_k \mid |\arg \frac{-\omega k}{a_{k+1}}| \le \varepsilon\}$$

and

$$|F^n(\tilde{z}) - \tilde{z}| = (|a_{k+1}\ k| + O(\varepsilon)) \cdot n \qquad .$$

from which we obtain

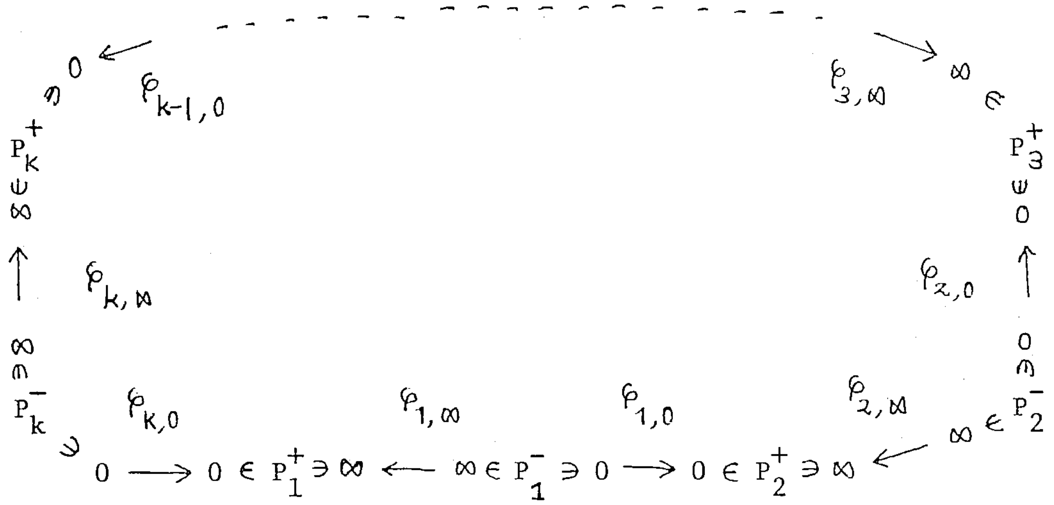
$$F^n(\tilde{z}) = \tilde{z} - n\ a_{k+1}\ k \ + \begin{cases} O(\log n) & k = 2 \\ O(n^{2-k}) & k > 2 \end{cases}$$

From now on we normalize $-a_{k+1}\ k = 2\pi\sqrt{-1}$. By the above form of F the quotient space P_i^+ (P_i^-) of the end of the upper (lower) half plane H_i^+ (H_i^-) of the i-th sheet of $\mathbb{C}_k$ by the lift F is homeomorphic to the two-sphere. These quotient spaces are quasi conformally homeomorphic hence conformally isomorphic to the doubly punctured Riemann sphere $P - 0 \cup \infty$ endowed with a coordinate z (unique up to scalar multiple). A fundamental domain $D_i^\pm$ in a half plane $H_i^\pm$ is *rectangular* if the boundary projects to a real line in $P_i^\pm$ joining 0 and ∞.

Later on we use the notations $P_{i+k}^\pm = P_i^\pm$ and $D_{i+k}^\pm = D_i^\pm$. The translation by an iteration of F of a large number of times carries both ends of D_i^- at 0, ∞ into H_{i+1}^+, H_i^+ respectively, hence induces germs of diffeomorphisms

$$\varphi_{i,0}: P_{i,0} \longrightarrow P_{i+1,0}, \qquad \varphi_{i,\infty}: P_{i,\infty} \longrightarrow P_{i,\infty} \quad ,$$

by transposing 0 and ∞ if necessary. These diffeomorphisms $\varphi = (\varphi_{i,j})_{i=1,\ldots,k,\ j=0,\infty}$ are displayed by the diagram



We define the equivalence relation: $\varphi \sim \varphi'$ if there exist $c_i^{\pm} \neq 0$ such that $\varphi'_{i,\infty}(c_i^- z) = c_i^+ \varphi_{i,\infty}(z)$ and $\varphi'_{i,0}(c_i^- z) = c_{i+1}^+ \varphi_{i+1,0}(z)$ for $i=1,\ldots,k$. It is easy to see that the equivalence class of φ is independent of the choice of the fundamental domains.

We now identify the union $D_i^+ \cup F(D_i^+) \cup F^2(D_i^+) \cup \ldots$ with the upper half plane H_i^+ and the union $D_i^- \cup F^{-1}(D_i^-) \cup F^{-2}(D_i^-) \cup \ldots$ with the lower half plane H_i^- on which F is the translation by $2\pi\sqrt{-1}$. With a sufficiently large n define the numbers $L_{i,0}$, $L_{i,\infty}$ by

$$\lim_{\tilde{z} \to j \in P_i^-} F^n(\tilde{z}) - \tilde{z} = L_{i,j} \quad , \quad j = 0, \infty .$$

Then we see that

$$L_{i,j} \equiv \log \varphi'_{i,j}(j) \mod 2\pi\sqrt{-1}.$$

Clearly these numbers depend on the choice of $D_i^{\pm}$ and the coordinates on $P_i^{\pm}$ and $H_i^{\pm}$. We now introduce the equivalence relation: $L = (L_{i,j}) \sim L' = (L'_{i,j})$ if there exist $c_i^{\pm} \neq 0$ such that

$L'_{i,0} = L_{i,0} + c^+_{i+1} - c^-_i$ and $L'_{i,\infty} = L_{i,\infty} + c^+_i - c^-_i$.
It is an easy exercise to show that the equivalence class of the tuple of the numbers $L(f) = (L_{i,j})$ corresponds to the invariant

$$m(f) = 1/2\pi\sqrt{-1} \quad \Sigma(L_{i,\infty} - L_{i,0})$$
$$(\equiv 1/2\pi\sqrt{-1} \quad \Sigma(\log \varphi'_{i,\infty}(0) - \log \varphi'_{i,0}(\infty)) \mod \mathbf{Z}).$$

we call $m(f)$ the *monodromy of* f or F. In particular $L(f)$ is equivalent to an L' such that $L'_{k,0} = (2\pi m(f)+m')\sqrt{-1}$ and $L'_{i,0} = L'_{i,\infty} = 2\pi m'\sqrt{-1}$ otherwise. In other words we may normalize $\varphi'_{i,j}(0) = 1$, $(i,j) \neq (k,0)$ by translating the fundamental domains $D^{\pm}_i$ and also $\varphi'_{k,0}(0) = 1$ by dividing $\varphi_{k,0}$ by $\varphi'_{k,0}(0)$. Respecting this normal form, the equivalence with $(c^{\pm}_i = c \neq 0)$: $\varphi = (\varphi_{i,j}) \sim \varphi' = (\varphi'_{i,j})$ if $\varphi_{i,j}(cz) = c\varphi'_{i,j}(z)$ is allowed.

Remark that by a calculation similar to that after Theorem 1.5, we see that a k-flat diffeomorphism is conjugate with a $g(z) = z + z^{k+1} + bz^{2k+1} + \ldots$ and formally conjugate with the $z + z^{k+1} + bz^{2k+1}$. By a straight forward argument with the form of the lift G of g, we see that the coefficient b determines the asymptotic behavior of G of order $\tilde{z}^i$, $i = 0, \pm 1$ at infinity, and $m(g)$ is a function of b (more precisely $m(g) = b - k+1/2$, cf. Theorem 1.9) and conversely the formal conjugacy class is determined by the residue. Thus the residue of holomorphic families of diffeomorphisms is holomorphic with respect to the parameter.

Conversely for given germs of diffeomorphisms $\varphi_{i,0}$ of P at 0 and $\varphi_{i,\infty}$ of P at ∞ , $i = 1,\ldots,k$ and a number $m(f)$ such that $2\pi\sqrt{-1}\ m(f) = \Sigma(\log \varphi'_{i,\infty}(0) - \log \varphi'_{i,0}(\infty)) + 2\pi m'\sqrt{-1}$ (log takes the principal value), we reconstruct the diffeomorphism f as follows.

Define the space Π by glueing the half spaces with handles

$$\tilde{H}^-_i = H^- \cup \{|\mathrm{Re}\ z| \geq M\}, \qquad \tilde{H}^+_i = H^+ \cup \{|\mathrm{Re}\ z| \geq M\}$$

by the relations $z \in \tilde{H}_i^- \sim z' \in \tilde{H}_i^+$ if $\tilde{\varphi}_{i,\infty}(z) = z'$ and $z \in \tilde{H}_i^- \sim z' \in \tilde{H}_{i+1}^+$ if $\tilde{\varphi}_{i,0}(z) = z'$. Here $\tilde{\varphi}_{i,0}$, $\tilde{\varphi}_{i,\infty}$ are representatives of the germs of diffeomorphisms $\log \varphi_{k,0} \exp + 2\pi m'\sqrt{-1}$, $\log \varphi_{i,0} \exp$, $i \neq k$ and $\log \varphi_{i,\infty} \exp$ covering $\varphi_{i,0}$, $\varphi_{i,\infty}$ respectively and M is a sufficiently large positive number such that both ends of the handles of $H_i^{\pm}$ are glued entirely with the opposite half planes. By Voronin [17] the germs of the resulting surface Π at infinity is quasi conformally homeomorphic and hence conformally isomorphic to the germ of the k-sheet covering $\tilde{\mathbb{C}}_k$ at infinity. The translations of $\tilde{H}_i^-$, $\tilde{H}_i^+$ by $2\pi i$ are compatible with the identifications and define a germ of a diffeomorphism F of Π, and hence a germ of diffeomorphism f of $\mathbb{C}$ at the origin.

It is known by Camacho [2] that k-flat diffeomorphisms $f(z) = z + az^{k+1} + \ldots$ are all topologically conjugate. It is then easy to see that the above constructed non linear translation F of the k-sheet covering is topologically conjugate with the translation by $2\pi\sqrt{-1}$ which is induced from the diffeomorphism $f_{2\pi\sqrt{-1}}(z) = (z^{-1/k} + 2\pi\sqrt{-1})^{-k}$ of $\mathbb{C},0$. Therefore the resulting diffeomorphism f has the form $f(z) = z + az^{k+1} + \ldots$, and by construction f has the given data φ and $m(f)$. The equivalence of data φ by $c_i^{\pm}$ $i = 1,\ldots,k$ induces the translations of $\tilde{H}_i^{\pm}$ by $\log c_i^{\pm}$, which induces a diffeomorphism of the surfaces Π. Therefore the diffeomorphism f is unique up to holomorphic conjugacy.

For a diffeomorphism f with the data $\varphi_{i,j}(f)$ and $m(f)$, choose the normal form $\psi'_{i,j}$ such that $\psi'_{i,j}(0) = 1$ for $(i,j) \neq (k,0)$ and put $\eta = \psi_{k,1}/\psi'_{k,1}(0)$. We define the germ of diffeomorphism f_0 with $m(f_0)$ associated to f by the germs $\psi_{i,j}$, $(i,j) \neq (k,0)$ and η. Conversely using $\psi_{i,j}$, $(i,j) \neq (k,0)$, η and $m(f)$, we can recover the initial data $\psi_{k,0}$, from which f is reconstructed.

Thus we obtain the following refinement of the well known theorem

Theorem 1.1 (Fatou, Kimura, Ecalle, Shcherbakov, Voronin, etc) *There exist one-to-one correspondences of the following sets.*

(1) *The set of holomorphic conjugacy classes of germs of* k-*flat diffeomorphisms of* $\mathbf{C},0$.

(2) *The set of equivalence classes of pairs* (φ,m): $(\varphi,m) \sim (\varphi',m')$ *if there exists a* $c_i^{\pm} \neq 0$ *such that* $\varphi'_{i,0}(c_i^{-}z) = c_{i+1}^{+}\varphi_{i,0}(z)$, $\varphi'_{i,\infty}(c_i^{-}z) = c_i^{+}\varphi_{i,\infty}(z)$ *for* $i = 1,\ldots,k$ *and* $m = m'$,

(3) *The set of equivalence classes of pairs* (φ,m) *with* $\varphi'_{i,j}(0) = 1$: $(\varphi,m) \sim (\varphi',m')$ *if there exists a* $c \neq 0$ *such that* $\varphi'_{i,j}(cz) = c\varphi_{i,j}(z)$ *for* $i = 1,\ldots,k$, $j = 0, \infty$ *and* $m = m'$.

(4) *The set of equivalence classes of the pairs* (f,m) *of* k-*flat diffeomorphisms* f *with* $m(f) = 0$ *and numbers* m: $(f,m) \sim (f',m')$ *if* f, f' *are conjugate and* $m = m'$.

Note that the above residu determines the formal conjugacy classes.

We apply the theorem to the d-iteration of f. Choose the fundamental domains $D_i^{\pm}(f^d) = D_i^{\pm}(f) \cup F(D_i^{\pm}(f)) \cup \ldots \cup F^d(D_i^{\pm}(f))$, $i = 1,\ldots,k$. F induces a diffeomorphism of the quotient spaces $P_i^{\pm}(F) = D_i^{\pm}(F^d)/F^d$ of order d, and the germs $\varphi_{i,j}(f^d)$ cover the germs $\varphi_{i,j}(f)$ via the quotients $P_i^{\pm}(f^d) \to P_i^{\pm}(f^d)/F = \varphi_i^{\pm}(f)$, given by $z \to z^d$ in a suitable coordinates by Lemma 1.4. Identifying $D_i^{\pm}(f^d)$ with the set of those $z \in H_i^{\pm}$ with $0 \leq \pm \operatorname{Im} z \lneq 2\pi\sqrt{-1}$ we obtain $L_{i,j}(f^d) = L_{i,j}(f)/d$ hence $d\, m(f^d) = m(f)$. Thus we obtain

Proposition 1.2 *$\varphi_{i,j}(f^d)$ are induced from $\varphi_{i,j}(f)$ via the map $z \to z^d$, and* $d\, m(f^d) = m(f)$.

Proposition 1.3 *Assume that $f(z) = z + az^{k+1} + \ldots$ and $g(z) = z + bz^{k+1} + \ldots$ commute. Then* $a\, m(f) = b\, m(g)$. *In general if $f^b = g^a$ then* $a\, m(f) = b\, m(g)$.

Proof. The second statement is proved above. If a, b are linearly dependent over $\mathbb{Z}$, the statement reduces to the second. The rest is the case where f, g embeds to a complexified one parameter family $\exp t\chi$ by Theorem 1.8. Then the statement follows from the fact $t\, m(\exp t\chi) = m(\exp \chi) = m$ for the χ in Theorem 1.8.

Lemma 1.4 [12] *Germs of diffeomorphisms of $\mathbb{C},0$ with order d are holomorphically conjugate with the linear rotation $\omega_d z$, ω_d being the primitive root of 1 of order d.*

Next we consider the centralizers C(f) of diffeomorphisms f. Consider the commutativity relation $f \circ g = g \circ f$ of diffeomorphisms $f(z) = z + az^{k+1} + \ldots$ and $g(z) = A_1 z + A_2 z^2 + \ldots$ of $\mathbb{C},0$. A direct calculation shows that the lowest order terms of the difference $f\circ g - g \circ f$ with the letter A_i in coefficient is

$$((k+1)a_{k+1}A_1^k A_i - ia_{k+1}A_i + R_{i+k})\, z^{i+k} ,$$

where R is a polynomial of $a_i, \ldots, a_{i+k}$ and $A_1, \ldots, A_{i-1}$. The commutativity implies that $A_1^k = 1$ and hence $(k+1-i)a_{k+1}A_i + R_{k+i} = 0$. This tells that the coefficients A_i are uniquely solved in terms of the proceeding terms except for the case $i = k+1$, and in particular if $A_1 = 1$ then $A_i = 0$ for $2 \le i \le k$. To achieve the formal commutativity the condition $R_{2k+1} = 0$ is required, and then the coefficient A_{k+1} can be arbitrary. The set of those A_1 for which $R_{2k+1} = 0$ form a cyclic subgroup $\mathbf{Z}_n \subset \mathbf{Z}_k$. A coefficient $A_1 \in \mathbf{Z}_n$ and an $A_{k+1} \in \mathbf{C}$ determine a formal diffeomorphism denoted $f_{A_1 A_{k+1}}$ and the composition raw of these diffeomorphisms induces a group structure on the set $\mathbf{Z}_n \times \mathbf{C}$, which is denoted $\tilde{C}(f)$. Denote the subgroup of $(A_1, A_{k+1}) \in \tilde{C}(f)$ for which the formal diffeomorphism converges by $C(f)$. We denote the kernels of the projections L of $\tilde{C}(f)$, $C(f)$ to the linear terms by $\tilde{C}^0(f)$, $C^0(f)$ respectively.

By easy algebra we have

Proposition 1.5 *The centralizers* $\tilde{C}^0(f)$, $C^0(f)$ *are commutative.*

Proposition 1.6 *Let* f *be* k-*flat. Then* $C(f) = C^0(f) \times \mathbf{Z}_n$ *with a divisor* n *of* k *if and only if* f *is conjugate with an* f', *which is factored by an* f'' *via the map* $z \to z^n$: $f'(z)^n = f''(z^n)$, n *being the largest of such numbers. And then* $\mathbf{Z}_n(f')$ *is generated by the linear rotation* $z \to \omega_n z$ *and* $C^0(f')$ *consists of* $g(z) = z + bz^{k+1} + \dots$ *with* b *in a closed subgroup* $\Lambda \subset \mathbf{C}$, *which are factored through* z^n.

Theorem 1.7 *Let* f *be* k-*flat and* $C(f)/C^0(f) = \mathbf{Z}_n$. *If* f *admits the* k-*flat holomorphic diffeomorphism* $f^{1/n}$ *of* $\mathbf{C},0$: $(f^{1/n})^n = f$, *then* $C(f) = C^0(f) \times \mathbf{Z}_n$.

The following theorem is known.

Theorem 1.8 (Baker [1,6]) $C^0(f)$ *is either a sequence* $c\mathbb{Z}$, c *rational or* $\mathbb{C}$. *If* $C^0(f) = \mathbb{C}$, *there exists a germ of diffeomorphism* φ *of* $\mathbb{C},0$ *such that* $\varphi f_{1b} \varphi^{-1} = \exp b\chi$, *where* χ *is the vector field in Theorem* 1.9.

Theorem 1.9 (see cf. [10,13]) *A* k-*flat holomorphic vector field* χ' *on* $\mathbb{C},0$ *is conjugate with the following normal form*

$$\chi(z) = \frac{z^{k+1}}{1 - m z^k} \partial/\partial z \quad .$$

We define the *monodromy* $m(\chi')$ of χ' by the above m. In other words $m(\chi')$ is defined by $(\exp 2\pi k\sqrt{-1}\chi')(z) = (\exp m(\chi')2\pi\sqrt{-1}\ \chi')(z)$. It is easy to see that $m(\exp t\chi) = m(t\chi) = m(\chi)/t = m/t$.

2. Commutative groups and Solvable groups

Assume that a commutator $[G,G]$ of a group G of diffeomorphisms of $\mathbb{C},0$ is commutative and consisting of k-flat diffeomorphisms. For a k-flat $f \in [G,G]$ and an i-flat $g \in G$ an easy calculation shows that $f^{-1}g^{-1}fg$ is j-flat, i, k $\lneq$ j. Since the diffeomorphisms of the commutative group $[G,G]$ are determined by their k+1-th order terms, (see Section 1) we have $fg = gf$, and $g(z) = z$ if $i \neq k$. This observation tells that the subgroup G^0 consisting of those $f \in G$ with the linear term z is commutative and k+1-flat, and the k-jets of $f \in G$ are determined by their linear terms and the following higher terms are determined only with the k+1-th order terms. In particular G is solvable if and only if meta-Abelian, that is, $[G,G]$ is commutative, and then the projections L, L^{k+1} of G/G^0, G^0 to the linear and the k+1-th order terms are injective homomorphisms into $\mathbb{C}$, $\mathbb{C}^*$ respectively. We denote those images by $L(G) \subset \mathbb{C}$ and $\Lambda(G) \subset \mathbb{C}^*$ respectively.

Now assume that G is solvable. Since G^0 is commutative, G^0 embeds to a complexified one parameter family of formal diffeomorphisms($= \tilde{C}^0(h_0)$, $h_0 \in G$) consisting of $h = h_0^{\lambda}(z) = z + 0 + \dots + 0 + \lambda b z^{k+1} + \dots$, $\lambda \in \mathbb{C}$ by a result in §1 (where h_0^{λ} is convergent for some λ). The action of G/G^0 on G^0 well defined by $\mu(g,h) = g^{-1}hg$, $g \in G/G^0$, is then represented as

(*) $\mu(g,h) = h^c$, $c = g'(0)^k$.

This equation admits unique formal solution g for given linear and k+1-th order terms and an h. By a suitable coordinate transformation we may assume that a diffeomorphism $g \in G$ has the form $g(z) = az + 0 + \dots + 0 + bz^{k+1} + \dots$. We show that <u>the other diffeomorphisms in G have then the same form</u>. The above g is formally linearized to the map $g_a(z) = az$, which admits the complex iteration $g_a^{\mu}(z) = a^{\mu}z$, $\mu \in C$. Therefore the g admits the formal complex iteration g^{μ} with which the compositions $h'g^{\mu}$ with the k-flat formal diffeomorphisms $h' \in \tilde{C}^0(f)$ produce all formal solutions of the equation (*), h being fixed. In particular we see that all diffeomorphisms $g \in G$ have the required form. Then a direct calculation shows that G <u>embeds, by the projections to the linear and k+1-th order terms, to the semi-direct product of $\mathbb{C}^*$ and $\mathbb{C}$ with the action</u> $\mu(a,b) = a^k b$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$.

By definition G is a central extension of L(G) by ker L if and only if the action μ is trivial if and only if $L(G) = \mathbb{Z}_n \subset \mathbb{Z}_k$ if and only if G is commutative.

Theorem 2.1 (Solvable groups) *Let* G *be a solvable group of germs of diffeomorphisms of* $\mathbb{C},0$. *Then* G *embeds to the semi-direct product*

$\mathbb{C}^* \times \mathbb{C}$ *with the action* $\mu(a,b) = a^k b$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$.

(0) G *is commutative if and only if the action* μ *is trivial*

(1) *If the action* μ *of* G/G^0 *on* G^0 *is non trivial and* $\Lambda(G) \subset \mathbb{C}$ *is non degenerate* ($\Lambda(G)$ *spans* $\mathbb{C}$ *over* $\mathbb{R}$), *then the action of* G *is conjugate with the following:*

$$G^0 = \{ \exp t\chi \ , \ t \in \Lambda(G) \}, \qquad \chi(z) = z^{k+1}\, \partial/\partial z \quad ,$$

and G/G^0 *consists of the classes of*

$$g(z) = b \cdot \exp c(b)\chi \qquad , \quad b \in L(G) .$$

If $b^k = 1$ *then* $g^k = \exp kc(b)\chi \in G^0$ *so* $kc(b) \in \Lambda(G)$. (*If* $b^i = 1$, $i \neq k$ *then* $g^i(z) = z$.)

(1)' *If* G^0 *embeds to a flow* $\exp t\xi$, *then* $m(\xi) = 0$ *or* G *is commutative*.

(2) *Assume* $\Lambda(G) = c\mathbb{Z}$, $c \neq 0$. *Then* $L(G) = \mathbb{Z}_n$, $k = n/2, n, 2n, 3n, \ldots$. *If* $k = n, 2n, 3n, \ldots$, *then* G *is commutative*. *If* $n = 2k$, G *is generated by a* k-*flat* f *and a* g *with the linear term* ω_{2k} *with the relation* $g^{-1}fg = f^{-1}$. *And then* $m(f) = 0$, $\varphi_{i0}(f) = \varphi^{-1}_{i+1\infty}(f) = \varphi_{i+10}(f)$ *for* $i = 1,\ldots,k$ ($\varphi_{k+1\infty}(f) = \varphi_{1\infty}(f)$) *and* $g = \omega_{2k} f^{p/q}$, *where* $f^{p/q}$ *is* k-*flat and* $(f^{p/q})^q = f^p$.

In Case (1) and (1)", we call an $g = \exp c\chi$, $c \in \Lambda(G)$ a *generator of* G^0 and denote $\exp d\chi = g^{d/c}$ for $d \in \Lambda(G)$.

Proof of (1) *and* (1)'. If $\Lambda(G)$ is non degenerate, G^0 embeds to a flow $\exp t\chi$ by Theorem 1.8. Let $f = \exp a\chi$ be a generator of G^0 and $g^{-1}fg = f^{\alpha}$. Since the residue is invariant under coordinate transformations we have $m(g^{-1}fg) = m(f)$, while we have

$m(f^\alpha) = m(f)/\alpha$ by Proposition 1.3. If $\alpha = 1$ for all $g \in G$, G is commutative. Otherwise we have $m(f) = m(\chi)/a = 0$ and hence $\chi(z) = z^{k+1}\,\partial/\partial z$. Let $g'(0) = b$. Since $b^{-1}f(bz) = \exp ab^k\chi(z) = f^{(b^k)}(z) = g^{-1}fg(z)$, we see that $b^{-1}g$ commutes with f and hence $b^{-1}g = f^c$ hence $g = bf^c = b \exp ca\chi$.

Proof of (2). Assume the action of L(G) on G^0 is non-trivial. Then clearly $n = 2k$ and $g^{-1}fg = f^{-1}$. Here f is the generator of G^0 and g has the linear term ω_{2k}. With rectangular fundamental domains $D_i^{\pm}(f)$ such that $g(D_i^-) = D_{i+1}^+$, $g(D_i^+) = D_{i+1}^+$, $g(t) = 1/t$, we have $\varphi_{i0}(f) = \varphi_{i+1\,\infty}^{-1}(f) = \varphi_{i+1\,0}(f)$ for $i = 1,\ldots,k$. The rest follows from the same argument as the other case.

Theorem 2.2 (Commutative groups) *Let* G *be a commutative group of germs of holomorphic diffeomorphisms of* $\mathbb{C},0$.

(1) *If the linear term* $L(G) \subset \mathbb{C}^*$ *contains either an* a, $|a| \neq 1$ *or* $a = \exp 2\pi\alpha\sqrt{-1}$, α *a Brjuno number* (*see* [2]), *then the homomorphism* L *is injective and* G *is linearisable*.

(2) *If* $L(G) = z_n$ *and* $G^0 \neq \mathbb{Z}$, *then* G^0 *is* nj-*flat*, G *is conjugate with a group generated by* $\exp a\chi$, $a \in \Lambda(G) \subset \mathbb{C}$ *and* $\omega_n \exp c\chi$, *where* χ *is a holomorphic vector field*.

(2)' *If* G^0 *embeds to a flow* $\exp t\chi$, *the same as* (2) *holds*.

(3) *Assume* $L(G) = \mathbb{Z}_n$, $\ker L = \mathbb{Z}$ *and let* f, g *be the generators of* G^0, G/G^0. *If* g^n *admits a flat* $g^{n/m}$, $n/m \in \mathbb{Z}$, *then* f *is factored through the covering* $z \to z^m$. *If* $g^n = f^\ell$ *and* ℓ/n *is an integer, then* L(G) *is generated by* $gf^{-\ell/n}$ *which is linearized*.

Proof of (1). By the Poincare linearization theorem, a diffeomorphism with a linear term a, $|a| \neq 1$ is conjugate with the linear function az. Then the commutativity of G implies that the other members are all linear. If exp $2\pi\alpha\sqrt{-1}$ and α is a Brjuno number , f is linearized ([2]) and then the same as above holds.

Proof of(2) and (2)'. By Theorem 1.8, after a coordinate transformation, G^0 embeds to a complexified flow exp $t\chi$. If g commutes with exp $a\chi$, g has the linear term ω_k^i. Clearly the linear rotation by ω_k^i commutes with exp $t\chi$, and $\omega_k^{-i}g$ = exp $c\chi \in G^0$. Thus $g = \omega_k^i$ exp $c\chi$. Denote the smallest of such i by j. Then k = nj and $L(G) = \mathbb{Z}_n$.

(3) follows from Lemma 1.4.

3 Construction of vector fields by commutators and preliminaries for the rigidity theorem

Lemma 3.1 *Let* X_{iu} , Y_{iu} , $u \in \mathbb{C}^r$, $i = 1,2$ *be holomorphic families of germs of non-singular vector fields of* $\mathbb{C},0$. *Assume that there exists a continuous family of germs of homeomorphisms* h_u *of* $\mathbb{C},0$ *such that* h_u exp tX_{iu} = exp tY_{iu} h_u . *If* $[X_{1u_0}, X_{2u_0}]$ *is not a real constant, then* h_u *is* $\pm$ *holomorphic with respect to* z *and* u *at* $0 \times u_0$. *And there exist only finitely many conjugacies* h_u.

Proof. First we assume that $r = 0, [X_1, X_2](0)$ is non-real and $X_1(0)$, $X_2(0)$ are independent over $\mathbb{R}$. It is easy to see that h is C^∞ smooth. So we have only to show the analyticity. The mapping $(s,t) \to$ exp $sX_1 \circ$ exp $tX_2(0)$ of $\mathbb{R}^2$ to $\mathbb{C}$ is by assumption non singular, so for small (s,t) there exists an $(s',t') = (s,t) + O^2(s,t)$ such that

$$\exp sX_1 \circ \exp tX_2(0) = \exp t'X_1 \circ \exp s'X_2(0) \text{ and}$$

$$D = d\,(\exp -s'X_1 \circ \exp -t'X_2 \circ \exp sX_1 \circ \exp tX_2)\,(0)$$

$$= id + st\,[X_1, X_2]\,(0) + O^3(s,t) \quad .$$

Define D' similarly with Y_1, Y_2. Since $[X_1,X_2](0)$ is non-real, D is so for small s, t, and $dh(0)\circ D = D'\circ dh(0)$ implies that $dh(0)$ is homothetic. The set of those z where $[X_1,X_2]$ is non-real is dense at the origin, on which this argument applies to say that h is ± holomorphic if $X_1(z)$, $X_2(z)$ are independent over $\mathbb{R}$.

Secondly we consider those z where X_1, X_2 are dependent over $\mathbb{R}$. The conjugacy $\exp tY_i \circ h = h \circ \exp tX_i$ tells that h is uniformly ± holomorphic along the trajectories of X_i if h is so at some point on it. The rest is the case where X_1, X_2 are dependent over $\mathbb{R}$ along the common trajectory C passing through the origin.

Now we have only to show the smoothness of h at the origin by choosing suitable coordinates. Let S be a smooth real curve transversal to C and define the diffeomorphism φ_t of $(S,0)$ by $\pi\circ(\exp tX_1|S)$ with the projection of $\mathbb{C}$ onto S along the trajectories of X_2. Since the ratio of the linear terms of X_1, X_2 with z is not constant along C (otherwise X_1, X_2 are $\mathbb{R}$-dependent on a neighbourhood of the origin), we see $\varphi_t'(0) \neq 1$ for $t \neq 0$. Clearly h makes φ_t conjugate with the diffeomorphism φ_t' of $(h(S),0)$ similarly constructed with Y_i. Here Proposition 3.2 applies to say that the restriction $h|S$ is a real analytic diffeomorphism. The smoothness of h follows from those of $h|C$ and $h|S$. This completes the proof of the analyticity of h_u.

By the hypothesis, X_{1u}, X_{2u} and Y_{1u}, Y_{2u} are linearly independent and X_{1u}/X_{2u}, Y_{1u}/Y_{2u} are non-zero non-constant holomorphic functions. Since h_u is ± holomorphic, we have $Y_{1u}/Y_{2u}\circ h_u = X_{1u}/X_{2u}$ or its complex conjugate, from which it follows h_u is $\pm$ holomorphic. This equality admits only finitely many (at most the branching degree of Y_{1u}/Y_{2u} at 0) solutions h_u.

Proposition 3.2 ([5]) *Let* f_i, g_i, $i = 1,2$ *be germs of* C^r *diffeomorphisms of* $\mathbb{R},0$ ($r = 2,3,\ldots,\infty, \omega$) *let* h *be a germ of homeomorphism of* $\mathbb{R},0$ *such that* $h f_i = g_i h$ *for* $i = 1,2$. *Assume that* $f_1'(0)^a f_2'(0)^b$, $a, b = 0, \pm 1, \pm 2, \ldots$ *are dense in* $\mathbb{R}$. *Then* h *is a germ of* C^r *diffeomorphism*.

Let G_u, G_u', $u \in \mathbb{C}^r$ be groups generated by holomorphic families of diffeomorphisms of $\mathbb{C},0$. If G_0 is non-solvable, G contains two diffeomorphisms of the forms $f_u(z) = z + az^{i+1} + \ldots$, $g_u(z) = z + bz^{j+1} + \ldots$, $i \not\leq j$, $a(0)$, $b(0) \neq 0$.
Then $[f_u, g_u](z) = z + cz^{k+1} + \ldots$ $j \not\leq k$, $c(0) \neq 0$. Let $f_u^n(z)$, $n = 0,1,\ldots$ be an orbit in the domain of the definition of f_u convergent to the origin. We will show that the dynamics $f_u^{-n} g_u f_u^n$ is convergent to the identity but a suitable scalar multiple $\lambda_n(f_u^{-n} g_u f_u^n - \mathrm{id})$ is convergent to an analytic vector field χ, which is well defined up to scalar multiplication. This vector field does not extend to a neighbourhood of the origin as $f_u^n(z) \not\to 0$ in general. (For such z, we use f_u^{-1} in place of f_u.) Using g_u and $[f_u, g_u]$ we define another dynamics ξ and prove that χ and ξ satisfy the condition of Lemma 3.1.

Using the coordinate $\tilde{z} = z^{-i}$ on the i-sheet covering $\mathbb{C}_i$ of the punctured neighbourhood of $\mathbb{C}$ at $\tilde{z}_0$, f_u is written as

$$F_u(\tilde{z}) = f_u(z^{-1/i})^{-i} = \tilde{z} - a\,i + A\,X^{-1/i} + A'\,X^{-2/i} + \ldots,$$

where $\tilde{z}$ takes the branch of $\tilde{z}_0$ and the F_u is defined at infinity. On the covering space the second diffeomorphism lifts to a very slow dynamics

$$G_u(\tilde z) = g_u(z^{-1/i})^{-i} = \tilde z - b\, i\, \tilde z^{-j/i} + B\, \tilde z^{-(j+1)/i} + \dots .$$

Our dynamics χ on $\mathbb{C}_i$ is defined by

$$\chi_u(\tilde z) = \lim_{n\to\infty} \lambda_n (F_u^{-n} G_u F_u^n - \mathrm{id})\, \partial/\partial\tilde z ,$$

with a sequence of real numbers $0 \le \lambda_n \to \infty$.

By the form of F_u, we have

$$(*)\qquad F_u^n(\tilde z) = \tilde z - n\, a\, i + \begin{cases} O(\log n) & i = 2 \\ O(n^{2-n}) & i > 2 \end{cases} .$$

First we show that dF_u^n is convergent to an analytic function. We begin with

$$\log dF_u^n = \log dF_u(F_u^{n-1}) + \log dF_u(F_u^{n-2}) + \dots + \log dF_u$$

$$= \sum_{j=0}^{n-1} \log\left(1 - A\, i\, (F_u^j)^{-(i+1)/i} - 2A\, i\, (F_u^j)^{-(2i+1)/i} - \dots\right)$$

where F_u^j denotes the j-fold iteration of F_u. Since $\sum_{j,k=0}^{\infty} (k+1)(F_u^j)^{-1/i\ -k-1}$ is locally uniformly convergent, dF_u^n is convergent to the dF_u^∞, which is holomorphic.

Lemma 3.3 dF_u^∞ *is a holomorphic function and* $dF_u^\infty(\tilde z) \to \mathrm{id}$ *as* $\tilde z \to \infty$.

Proof. $|\log dF_u^\infty|$

$$\le \sum_{j=0}^{\infty} \left|\log(1 - i\, A(F_u^j)^{-1/i\ -1} - 2\, i\, A'(F_u^j)^{-1/i\ -2} - \dots)\right|$$

$$\le \sum_{j=0}^{\infty} K \left| i\, A(F_u^j)^{-1/i\ -1} \right| ,$$

and since $F_u^j(\tilde z) = \tilde z - a\, n\, i + O(n^{i-1/i})$ (or $\tilde z - a\, n\, i + O(\log n)$),

$$\le K' \, |\tilde z|^{-1/i} ,$$

for $\tilde z$ such that $|F_u^j(\tilde z)|$, $j = 0, \dots$ are sufficiently large.

Let $\lambda_n = n^{j/i}$. Then we have

$$\lim_{n\to\infty} \lambda_n (F_u^{-n} G_u F_u^n - \mathrm{id}) = \lim_{n\to\infty} \lambda_n (dF_u^{-n}(G_u - \mathrm{id})F_u^n + O(F_u^n))$$

$$= \lim_{n\to\infty} dF_u^{-n}(-b\ i\ (n/F_u^n)^{j/i}) \text{ and by } (*) = dF_u^{-\infty}(\ -bi\ (-ai)^{j/i}),$$

by which we define the vector field χ. We may assume that $g^n(z_0) \to 0$ (otherwise use g^{-1}). Similarly we have

$$\lim_{n\to\infty} n^{k/j}(G_u^{-n}[F_u, G_u]G_u^n - \mathrm{id}) = dG_u^{-\infty}(\ -cj\ (-ai)^{k/j})$$

on the covering space $\mathbb{C}_i$, by which we define another dynamics μ.

Lemma 3.4 χ, μ *are nowhere zero holomorphic vector fields and linearly independent over* $\mathbb{R}$.

Proof. We assume that $r = 0$ and $\mu = a\chi$ with a constant $a \neq 0$. Since μ is invariant under dG, χ is also invariant. On the other hand we see that dG tends to 0 since g lifts to G' on the j-sheet covering $\mathbb{C}_j$ for which dG'^n converges to the linear map dG'^{∞}. This contradicts that $dG^n(\chi) = \chi(G^n) \to 0$ while $\chi(G^n) = dF^{-\infty}(G^n)(\ -bi\ (-ai)^{j/i}\ )$ is convergent to a non-zero constant by Lemma 3.3.

Lemma 3.5 $[\chi,\mu]$ *is linearly independent of* χ, μ *and* 1.

Proof. We assume that $r = 0$ and $[\chi,\mu] = a + b\chi + c\mu$, with constants a, b and c. Since $dF^{-\infty}(G^n) \to \mathrm{id}$, $dG^n \to 0$ and χ, μ are invariant under dF, dG, respectively, we see that $a = b = 0$. We then solve the equation $[\chi,\mu] = c\mu$ as $\mu = d\ \chi \exp D\int 1/\chi$. Since χ converges to $-bi\ (-ai)^{j/i}\partial/\partial\tilde{z}$, μ has the order of $\exp D\tilde{z}$. On the other hand μ is convergent to a non-zero constant vector on the j sheet covering space $\mathbb{C}_j$ at infinity so μ has the order of

$\tilde{z}^{i-j}$. This is a contradiction.

Proposition 3.6 *The real vector fields χ, μ are invariant under topological conjugacy.*

Proof. Let G, G' be non-solvable groups acting on $\mathbb{C},0$ and h be a topological conjugacy: there exists a group isomorphism $\varphi:G \to G'$ such that $g\circ h = h\circ\varphi(g)$ for $g \in G$. Construct the vector fields χ', μ' similarly with $f' = \varphi(f)$ and $g' = \varphi(g)$. Define the vector fields χ_n by $\chi_n(\tilde{z}) = n^{j/i}(F^{-n}GF^n - \mathrm{id})\,\partial/\partial\tilde{z}$. Since χ_n, $\chi_\infty = \chi$ are smooth and $\chi_n \to \chi$ as $n \to \infty$, the real trajectories $\exp t\chi_n(\tilde{z}_0)$, $t \in \mathbb{R}$ passing though $\tilde{z}_0$ are arbitrary well approximated by the sequence $\tilde{z}_{\ell+1} = t_\ell\chi_n(\tilde{z}_\ell)$, $\ell = 0,1,\dots,m-1$ with sufficiently small t_ℓ, $\Sigma\, t_\ell = t$. Then the difference $|\exp t\chi_n(\tilde{z}_0) - \tilde{z}_m|$ for $\tilde{z}_0$ in a compact set in $\mathbb{C}_i$, $0 \le t \le a$, $n = 1,2,\dots,\infty$, has a uniform upper bound $C(\delta)$ depending only on $\delta = \max\{t_\ell\}$ such that $C(\delta) \to 0$ as $\delta \to 0$. Let $t_\ell = n^{-j/i}$ and $t^\ell = \ell\, n^{-j/i}$ with a sufficiently large n. Then by definition we have

$$\tilde{z}_{\ell+1} = \tilde{z}_\ell + n^{-j/i}\chi_n(\tilde{z}_\ell) = F^{-n}\,G\,F^n(\tilde{z}_\ell) = F^{-n}\,G^\ell\,F^n(\tilde{z}_0) \text{ for } \ell$$

$= 0,1,\dots$ and the above argument says that

$$|\exp t^\ell\chi_n(\tilde{z}_0) - F^{-n}G^\ell F^n(\tilde{z}_0)| \le C(n^{-j/i})$$

for $0 \le t^\ell \le a$. On the other hand the real trajectory $\exp t\chi_n(\tilde{z}_0)$, $t \in \mathbb{R}$ converges to $\exp t\chi(\tilde{z}_0)$. Any t, $0 \le t \le a$ is approximated by a $t^{\ell_n} = \ell_n\, n^{-j/i}$, for which $F^{-n}G^{\ell_n}F^n(\tilde{z}_0) \to \exp t^{\ell_n}\chi_n(\tilde{z}_0) \to \exp t\chi(\tilde{z}_0)$ as n tends to infinity. The conjugacy h lifts to a homeomorphism $\tilde{h}$ of the i-sheeted covering $\mathbb{C}_i$, which maps the orbits $F^{-n}G^\ell F^n(\tilde{z}_0)$ to the orbits defined with f' and g'. Thus the $\tilde{h}$ maps the real trajectories of χ to those of χ'. The same argument holds for μ and μ'.

4 The topological rigidity theorem

Let G_u, G'_u , $u \in V \subset \mathbb{C}^r$ be groups generated by holomorphic families of germs of diffeomorphisms of $\mathbb{C},0$ with the parameter u in an open neighbourhood V of the origin in $\mathbb{C}^r$. We say G_u, G'_u are *topologically conjugate* if there exists a continuous family of germs of homeomorphism h_u of $\mathbb{C},0$ and isomorphisms $\varphi_u\colon G_u \to G'_u$ such that $\varphi_u(f_u)$ is a holomorphic family in G'_u for generators $f_u \in G_u$ and $hf = \varphi(f)h$ holds, that is, the following diagram commutes

$$\begin{array}{ccc} f_u\colon \mathbb{C},0 & \longrightarrow & \mathbb{C},0 \\ h_u \downarrow & & \downarrow h_u \\ \varphi_u(f_u)\ :\ \mathbb{C},0 & \longrightarrow & \mathbb{C},0 \end{array}$$

for $f_u \in G$. Then there exists a homeomorphism $\tilde{h}\colon U \to \tilde{h}(U)$ of neighbourhoods of $0 \times V$ in $\mathbb{C} \times V$ representing h such that $\tilde{h}f_u = \varphi(f_u)\tilde{h}$ holds on an neighbourhood of the origin in U (depending on f_u) We call $\tilde{h}$ as well as h a *linking homeomorphism*.

Theorem 4.1 (**Topological rigidity theorem**) *A linking homeomorphism* h_u *is* $\pm$ *holomorphic with respect to* z *and* u *at the origin if one of Conditions* (1)-(3) *holds*.

(1) G_0 *is non-solvable, that is,* $[G_0,G_0]$ *is not commutative.*

(2) G_0 *is non-commutative but solvable,* $[G_0,G_0] = \Lambda(G_0) \subset \mathbb{C}$ *is dense and the action of* $L(G_0) \subset \mathbb{C}^*$ *on* $\Lambda(G_0)$ *contains a non-real multiplication*.

(3) G_0 *is non-commutative but solvable and the action of* $L(G_0)$ *on* $\Lambda(G_0)$ *contains the linear rotation by* $\mathbb{Z}_n$, $n \neq 2,3,4$, *or a non-periodic action*.

(4) *There exists a germ of holomorphic diffeomorphism linking* G *and* G' *if* G_0 *is non commutative but solvable and the action of* $L(G_0)$ *on* $\Lambda(G)$ *is not antipodal*.

(5) *There exists a germ of real analytic diffeomorphism linking* G *and* G' *if* $L(G) = 1$, G *is* k-*flat and* $\Lambda(G) \subset \mathbb{C}$ *is non degenerate*.

Proof of (1). For u sufficiently close to the origin G_u is non- solvable. First we consider the case $r = 0$. By the classification of solvable group (Theorem 2.1), the commutator subgroup $[G,G]$ contains non-commutative f, g. Let χ, λ be the holomorphic vector fields constructed in Section 3. The construction uses the diffeomorphisms f, g and the commutator $[f,g]$, so we assume there exists a homeomorphism $\tilde{h}: U \to \tilde{h}(U)$ of open neighbourhoods U, $\varphi(U)$ of the origin $0 \in \mathbb{C}$, on which f, g and $\varphi(f)$, $\varphi(g)$ are defined respectively and $\varphi(f)\tilde{h} = \tilde{h}f$ and $\varphi(g)\tilde{h} = \tilde{h}g$ hold as long as these maps are defined. From now on we denote $\tilde{h}$ simply by h. Define vector fields χ', λ' on an open subset of $\varphi(U)$ with $\varphi(g)$, $\varphi(g)$. By Proposition 3.6, $\varphi \exp t\chi = \exp t\chi' \varphi$ and $\varphi \exp t\lambda = \exp t\lambda' \varphi$. By Lemmas 3.1,3.4 and 3.5 h is $\pm$ holomorphic at those points $z \in U$ where χ, λ are defined and such that χ', λ' are defined at $\tilde{h}(z)$. For z sufficiently close to the origin, the forward or backward orbits of f, g converge to the origin, with which the vector fields are defined. Thus h is $\pm$ holomorphic on a punctured neighbourhood of the origin in U, and by Riemann's extension theorem, ± holomorphic at the origin. When $r \neq 0$, h_u is ± holomorphic with respect to u by Lemmas 3.1 and 3.4.

Proof of (2). First we assume $r = 0$. We use the notations in Section 1. By Theorem 1.3, G^0 consists of the diffeomorphisms $\exp t\chi$, $t \in \Lambda(G)$ with the holomorphic vector $\chi(z) = z^{k+1}\,\partial/\partial z$, which lift to the translations by t on the k-sheet covering space $\tilde{\mathbb{C}}_k$. The lift h' of the topological equivalence h is a conjugacy of the translations of $\tilde{\mathbb{C}}_k$ by $\Lambda(G)$ and $\Lambda(G')$, and hence real linear isomorphism and induces the isomorphism φ of $\Lambda(G)$ and $\Lambda(G')$.

Let $g(z) = bz + \ldots \in G$ such that b^k is non-real. Then the actions $f \to g^{-1}fg$, $f' \to \varphi(g)^{-1}f'\varphi(g)$ induce the linear multiplications by b^k, b'^k on $\Lambda(G)$, $\Lambda(G')$, which are conjugate by φ. Thus φ is homothetic, and hence the h is so.

We consider the case $r \neq 0$. The commutativity of $[G,G]$ reduces to that of the commutators of the generators. So G_u is solvable for all u, or the subset K of u for which g_u is solvable is contained in an analytic variety of $\mathbb{C}^r$. By (1) of the theorem, h is holomorphic at $0 \times u'$, $u' \notin K$. Therefore if the solvability does not hold in general, h is holomorphic by the Hartogs' continuous extension theorem. So we assume that g_u is solvable for all u. Then the density of $\Lambda(G_u)$ holds for all u and h induces a continuous family of real linear conjugacies of the translations of $\tilde{\mathbb{C}}_k$ by $\Lambda(G_u)$ and $\Lambda(G'_u)$. Since φ maps holomorphic families in G to those in G', h is holomorphic with respect to u.

Proof of (3). Assume that G_0 is non-commutative but solvable. Then the subgroup G_0^0 is isomorphic to $\Lambda(G_0) \subset \mathbb{C}$, which is invariant under the action of L(G). If the action contains the linear rotation $\mathbf{Z}_n$, $n \neq 2,3,4$ or a non-periodic action, then $\Lambda(G_0)$ is dense Therefore the proof reduces to Case (2).

Proof of (4). Assume that G_0^0 consists of $\exp t\chi$, $t \in \Lambda(G_0) \subset \mathbf{C}$ and G_0/G_0^0 is generated by $b \exp c(b)\chi$, $b \in L(G_0)$, where χ is the normal form in Theorem 1.9. If $\Lambda(G_0) \subset \mathbf{C}$ is dense, the rigidity holds by (2). Since the action of $L(G_0)$ is not antipodal, if $\Lambda(G_0)$ is not discrete, it is dense. So assume that $\Lambda(G_0)$ is discrete, that is, a non-degenerate lattice $\lambda\mathbf{Z} + \mu\mathbf{Z}$, on which $L(G_0)$ acts through the complex multiplication. By Theorem 2.1, the quotient G_0/G_0^0 is generated by a $g = \omega_n \exp c\chi$, which is linearized by $\exp d\chi$, $d = (b^k-1)^{-1}$: $\exp -d\chi(\omega_n \exp d\chi) = \omega_n \exp c\chi$. And then the subgroup G_0^0 remains the same form. Assume that G_0, G_0' are topologically conjugate. Then $G_0'^0$ is also k-flat and the actions of G_0^0, $G_0'^0$ on $\mathbf{C}_k$ are topologically conjugate. So $\Lambda(G_0')$ is also non-degenerate. The conjugacy tells also that the linear terms of G_0, G_0' coincide and their actions on G_0/G_0^0 , $G_0'/G_0'^0$ are algebraically conjugate. Since the actions are not antipodal, $\Lambda(G_0)$ and $\Lambda(G_0')$ are isomorphic. Therefore G_0, G_0' are holomorphically conjugate. If G_u satisfies one of Conditions (1)-(3) for generic u, the rigidity holds by the Hartogs' continuous extension theorem. Otherwise Condition (4) holds for all u close to the origin. Then G_u is holomorphically trivial.

Proof of (5). Assume that G consists of $\exp t\chi$, $t \in \Lambda \subset \mathbf{C}$, where Λ is a non degenerate subgroup and χ is a k-flat vector field of the normal form in Theorem 1.9. Then the residue $m(\chi)$ is given by $[f^n,g^n]^k = \exp m(\chi)\chi$ with an anti-clockwise k-fold rotation by f, $g \in G$. Let G' consists of $\exp t\chi'$, $t \in \Lambda'$ and assume that G, G' are topologically conjugate. Then the conjugacy induces an isomorphism of Λ and Λ' which extends to $\mathbf{C}$ real linearly and corresponds $m(\chi)$ to $m(\chi')$. The extension lifts to the k-sheet covering $\mathbb{C},0$ and then induces a germ of real analytic diffeomorphism of $\mathbb{C},0$, by which G, G' are conjugate. The case $r \neq 0$ follows those in Case (1)-(4).

5 The rigidity theorem for analytic correspondences

Let $\Gamma_i \subset \mathbb{C} \times \mathbb{C}$, $i = 1,2,\ldots$ be germs of analytic correspondences of type (p_i,q_i): germs of analytic curves with the intersection multiplicity p_i, q_i with $0 \times \mathbb{C}$, $\mathbb{C} \times 0$ respectively.

We say that Γ_1, Γ_2 are *equivalent* if there exist germs of holomorphic diffeomorphisms h_1, h_2 of $\mathbb{C},0$ such that $h_1 \times h_2(\Gamma_1) = \Gamma_2$. We say that Γ_1, Γ_2 are *topologically equivalent* if the above h_i are germs of homeomorphisms.

Let $P:\tilde{\Gamma}_i \to \Gamma_i$ be the normalizations. The composition $\pi_j = P_j P$ with the j-th projection of $\mathbb{C} \times \mathbb{C}$ are branched coverings with p_i, q_i sheets. The groups $G(\Gamma_i)$ are generated by the residue actions of orders p_i and q_i acting on $\tilde{\Gamma}_i$. We are interested in the classification of correspondences Γ via that of the groups $G(\Gamma)$.

A <u>diagram Γ of correspondences</u> is a collection of correspondences Γ_i, $i = 1,\ldots,k$, which we display as follows

$$\Gamma: \quad \begin{array}{l} \Gamma_1:\mathbb{C} \searrow \\ \Gamma_2:\mathbb{C} \longrightarrow \ \mathbb{C} \\ \quad\vdots \\ \Gamma_k:\mathbb{C} \nearrow \end{array}$$

We say two diagrams $\Gamma = \{\Gamma_i\}$, $\Gamma' = \{\Gamma'_i\}$ are <u>equivalent</u>

(<u>topologically</u> <u>equivalent</u>) if there exist germs of diffeomorphisms (homeomorphisms) h_i of $\mathbb{C},0$ for $i = 0,\dots,k$ such that $(h_i \times h_0)(\Gamma_i) = \Gamma'_i$ for $i = 0,\dots,k$. For a given diagram Γ we define the diagram of germs of holomorphic maps

$$\pi:\quad \begin{tikzcd} \mathbb{C} & \tilde{\Gamma}_1 \arrow[l, "\pi_{11}"'] \arrow[rd, "\pi_{12}"] & \\ \mathbb{C} & \tilde{\Gamma}_2 \arrow[l, "\pi_{21}"'] \arrow[r, "\pi_{22}"] & \mathbb{C} \\ \vdots & \vdots & \\ \mathbb{C} & \tilde{\Gamma}_k \arrow[l, "\pi_{k1}"'] \arrow[ru, "\pi_{k2}"'] & \end{tikzcd}$$

by the compositions $\pi_{ij} = P_j P$ of the projections and the normalizations $P : \tilde{\Gamma}_i \to \Gamma_i$. The classification of the diagrams Γ reduces to the classification of the diagrams of map germs π by the natural conjugacy. The projection g of the fibre product $\tilde{\Gamma}$ of the π_{i2} , $i = 1,\dots,k$ onto the right $\mathbb{C}$ has the sheet number $L =$ the least common multiple (LCM) of q_i and the compositions f_i of the projections of $\tilde{\Gamma}$ onto $\tilde{\Gamma}_i$ with π_{i1} have the sheet numbers $p_i \times L/q_i$. We display these projection as follows

$$\begin{tikzcd} f_1:\ \mathbb{C} & & \\ f_2:\ \mathbb{C} & \tilde{\Gamma} \arrow[lu] \arrow[l] \arrow[ld] \arrow[r, "g"] & \mathbb{C} \\ f_k:\ \mathbb{C} & & \end{tikzcd}$$

We define the group $G(\Gamma)$ acting on $\tilde{\Gamma}$ to be generated by the monodromy actions of the above projections.

Conversely let f_i, g be germs of diffeomorphisms of $\mathbb{C},0$ with order p'_i and L. Then the image Γ of the quotient maps $\mathbb{C} \to \mathbb{C}/f_i \times \mathbb{C}/g = \mathbb{C} \times \mathbb{C}$ is a germ of an analytic curve (Lemma 1.4). Then the first projection of Γ has p_i sheets and the quotient map (f_i,g) is a p'_i/p_i-sheet branched covering of the image. Here p_i is the least number such that $f^{p_i} = g^q$ with a q. Therefore (f_i,g) is the normalization if and only if $f^{p'} = g^{q'}$ implies $f^{p'} = \mathrm{id}$. The generators f_i, g of the group $G(\Gamma)$ have the orders $p_i \times L/q_i$, L and the relation $f_i^{p_i} = g^{q_i}$, which reconstruct the correspondences $\Gamma_i \subset \mathbb{C} \times \mathbb{C}$.

For a pair of correspondences Γ: $\mathbb{C} \overset{\Gamma_1}{\underset{\Gamma_2}{=\!=}} \mathbb{C}$, the above argument gives the diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[dd, equal] & & \\
& \tilde{\Gamma} \arrow[ul, "f_1'"'] \arrow[dl, "f_2'"] \arrow[r, "g"] & \mathbb{C} \\
\mathbb{C} & &
\end{tikzcd}$$

Then the composition $h = f_2' f_1'^{-1}$ "acts" on $\tilde{\Gamma}$, by which the actions φ, ψ by f_1', f_2' are *semi-conjugate*: $h\, f_1' = f_2'\, h$ (h is not a single-valued diffeomorphism). We denote by $\tilde{G}(\Gamma)$ the "group" generated by $G(\Gamma)$ and the "diffeomorphism" $f_2'\, f_1'^{-1}$.

Conversely by the $f_2'\, f_1'^{-1}$ we identify the quotient spaces $\mathbb{C}/f_1$, $\mathbb{C}/f_2$ and we reconstruct the diagram Γ.

We summarize the above relations of the diagrams and the groups by the dictionary:

$\Gamma \quad \vdots \gt \mathbb{C}$	$\Longleftrightarrow$	$G(\Gamma)$ finitely generated
Source spaces of f	$\Longleftrightarrow$	generators of finite order
identify the sources of f, g	$\Longleftrightarrow$	semi-conjugacy of generators

Let $\Gamma_u = (\Gamma_{ui})$, $\Gamma_u' = (\Gamma_{ui}')$, $u \in \mathbb{C}^r$, $i = 1,\ldots,k$ be diagrams of k holomorphic families of germs of analytic correspondences as above. We say Γ_u and Γ_u' are *equivalent* (respectively *topologically equivalent*) if there exist holomorphic families of germs of diffeomorphisms (continuous family of germs of homeomorphisms) h_{ui}, h'_u of $\mathbf{C},0$ such that $h_{ui} \times h'_u\,(\Gamma_u) = \Gamma'_u$ for $u \in \mathbf{C}^r$.

By the historical study initiated by Zarisky, etc, it is known that topologically trivial families of irreducible <u>curves</u> Γ_{ui} admit simultaneous resolutions $\pi_{ui}: \tilde{\Gamma}_{ui} \to \Gamma_{ui}$. If Γ_{ui} is of constant type (p,q), then the monodromy actions of the compositions $P_i\pi_{ui}$ are holomorphic with respect to u with order independent of u, which lift to the fibre product $\tilde{\Gamma}_u$ to generate the holomorphic family of groups $G(\Gamma_u)$ of diffeomorphisms of $\tilde{\Gamma}_u$ at 0.

Theorem 5.1 (Topological rigidity theorem for analytic correspondences) *Let Γ_u, $\Gamma'_u \subset \mathbf{C} \times \mathbf{C}$, $u \in \mathbf{C}^r$ be unions of holomorphic families of irreducible correspondences of constant type, which are topologically trivial as families of germs of curves. Assume that Γ_u, Γ'_u are topologically equivalent.*

(1) *The group $G(\Gamma_u)$ is commutative for an irreducible Γ_u if and only if Γ_u is equivalent to a $\Gamma = \{x^q = y^p\}$, p, q being coprime. Then $G(\Gamma_u)$ is linearizable and Γ_u, Γ'_u are holomorphically equivalent.*

(2) *If Γ_0, Γ'_0 are of type (1,p) or (p,1), then Γ, Γ' are holomorphically equivalent.*

If one of Conditions (3)-(7) holds, then the equivalences h_u, h'_u are $\pm$ holomorphic families of diffeomorphisms.

(3) *$G(\Gamma)$ is non-solvable for an irreducible component Γ of Γ_0.*

(4) *Γ_0 has an irreducible component Γ of type (p,q), p, q $\neq$ 1, (p,q) $\neq$ (2,2),(2,4),(4,2),(3,3),(4,4), for which $G(\Gamma)$ is non-commutative.*

(5) Γ_0 *has two irreducible components* Γ_1, Γ_2 *of types* $(i,i),(j,j)$ *with* $3 \leq i + j$, $1 \leq i,j \leq 4$, *such that the tangent lines satisfy a certain generic condition and* $G(\Gamma_k)$ *is non-commutative for* $k = 1$ *or* 2.

(6) Γ_0 *has three components of type* $(1,1)$, *which satisfy a certain generic condition*.

(7) *The L.C.M. of* p_i, q_i *for those* i *for which* $G(\Gamma_i)$ *is non-commutative is equal or greater than* 5.

(8) *If* Γ_0 *has an irreducible component* Γ *of type* $(2,4)$, $(4,2)$, $(3,3)$, $(4,4)$ *for which* $G(\Gamma)$ *is non-commutative, then* Γ_u, Γ'_u *are holomorphically equivalent*.

Proof of (1). Let Γ be irreducible of type (p,q) for simplicity. Then the group $G(\Gamma)$ is generated by two diffeomorphisms f, g with linear terms ω_p, ω_q and $L(G(\Gamma)) = \mathbb{Z}_{pq/n}$, n being the greatest common measure (GCM) of p, q. If f, g commute, then $\ker L = \phi$ and hence L is an isomorphism. By Lemma 1.4, f, g are linearized and the quotient map by f, g is (z^p, z^q) and the image is the curve $x^{p/n} = y^{q/n}$. Therefore we have $n = 1$.

Proof of (2). If Γ_0 is of type $(1,p)$, then Γ_u is a graph of holomorphic function f_u of $\mathbb{C},0$ with multiplicity p. It is then easy to see that f_u is holomorphically equivalent to the trivial family of the function z^p.

Proof of (3). Let Γ_u be irreducible and defined by a Puiseux expansion

$$F_u(z) = a_q\, z^{q/p} + a_{q+k}\, z^{q+k/p} + \ldots$$

and define

$$f_u(z) = F(z^p)^{1/q} = \sqrt{a_q}\; z\, (\, 1 + a_{q+k}/q\; z^k + \ldots \quad) \;,$$

where the coefficients are holomorphic functions of u. Then $(z^q, f_u(z)^p)$ normalizes the curve Γ_u, and $\tilde{\Gamma}_u$ is identified with the second coordinate axis. Put

$$f_{ij}(z) = \omega_p^i f(\omega_q^j z)$$

for $i = 1, \ldots, p$, $j = 1, \ldots, q$, where ω_p, ω_q are the primitive roots of order p, q.

The group $G(\Gamma_u)$ is generated by

$$\begin{aligned} f_{ij\alpha\beta}(z) &= f_{ij} f_{\alpha\beta}^{-1}(z) \\ &= \omega_p^{i-\alpha}\ \omega_q^{j-\beta}\ z\ (\ 1 + a_{q+k}/q\ \omega_p^{-\alpha k}(\omega_q^{(j-\beta)k} - 1)\ z^k + \ldots\ ) \end{aligned}$$

$i, \alpha = 1, \ldots, p$, $j, \beta = 1, \ldots, q$.

The homeomorphisms h_u, h_u' lift to those $\tilde{h}_u$, $\tilde{h}_u'$ of the p-sheet, q-sheet coverings of $\mathbb{C}, 0$, by which the functions f_u, f_u' are conjugate, and in particular the groups $G(\Gamma_u)$, $G(\Gamma_u')$ are conjugate by $\tilde{h}_u$. So if $G(\Gamma_0)$ is non-solvable, $\tilde{h}_u$ and $\tilde{h}_u'$ are ± holomorphic with respect to $\tilde{z}$ and u and hence h_u, h_u' are also ± holomorphic.

Proof of (4). We assume that $\Gamma_0 = \Gamma$ is irreducible, $G(\Gamma_0)$ is solvable, and analyse the commutator group by using the coefficients of the expansion of the above f.

Before we begin the detailed argument we recall some results on solvable groups of diffeomorphisms in general from Section 2. By Theorem 2.1, the action of a solvable group G is conjugate with the following: G^0 is generated by an $f = \exp a\chi$, $\chi(z) = z^{k+1}\, \partial/\partial z$ or a k-flat diffeomorphism with $m(f) = 0$, G/G^0 is generated by $\omega_n f^{p/q}$, n being a factor of k (for the detail, see Section 2). The conjugacy of G to the the above normal form induces that of the quotient spaces of $\mathbb{C}$ by the generators of $G(\Gamma)$, and deforms Γ to a Γ' for which $G(\Gamma')$ has the form. So we may assume $G(\Gamma)$ has the above normal form. The generators $\omega_n f^{p/q}$ then has the form

$$\omega_n\ f^{p/q}(z) = \omega_n z + p/q\ a\ \omega_n\ z^{k+1} + \ldots$$

and the k+1-th order term of G^0 does not vanish.

By a direct calculation, we have

$$(*)\quad [f_{ij\alpha\beta}, f_{i'j'\alpha'\beta'}](z)$$

$$= z + a_{q+k/q}\{\omega_p^{-\alpha k}\omega_p^{i-\alpha}\omega_q^{j-\beta}(\omega_q^{(j-\beta)k}-1)(1-\omega_p^{(i'-\alpha')k}\omega_q^{(j'-\beta')k})$$

$$- \omega_p^{-\alpha' k}\omega_p^{i'-\alpha'}\omega_q^{j'-\beta'}(\omega_q^{(j'-\beta')k}-1)(1-\omega_p^{(i-\alpha)k}\omega_q^{(j-\beta)k})\}\, z^{k+1} + \dots$$

If $\omega_q^k = 1$, the k+1-th order terms vanish and $[G,G] = 0$ and hence G is commutative by the above assumption. So we assume $\omega_q^k \neq 1$.

We claim that $\omega_q^{(j-\beta)k} \neq 1$ if $\omega_q^{j-\beta} \neq 1$. So assume that the equality holds. Assume that $\omega_q^{(j-\beta)k} = 1$. Then we have

$$f_{ij\alpha\beta}(z) = \omega_p^{i-\alpha}\ \omega_q^{j-\beta}\ z\ (\ 1 + O(z^{k+1})\ )$$

from which with the assumption

$$f_{ij\alpha\beta}(z)^n = z + O(z^{k+1}) = z \qquad .$$

By Lemma 1.4 we may assume

$$f_{ij\alpha\beta}(z) = \omega_p^{i-\alpha}\ \omega_q^{j-\beta}\ z$$

from which we have

$$f_{ij}(z) = \omega_p^i\ f(\omega_q^j\ z) = \omega_p^{i-\alpha}\ \omega_q^{j-\beta}\ f_{\alpha\beta}(z) = \omega_p^i\ \omega_q^{j-\beta}\ f(\omega_q^\beta\ z)$$

hance

$$f(\omega_q^{j-\beta}\ z) = \omega_q^{j-\beta}\ f(z) \qquad .$$

This shows that

$$f(z) = a_q\ z^{q/p} + a_{q+k'}\ z^{q+k'/q} + a_{q+2k'}\ z^{q+2k'/q} + \dots$$

k' being the least number with $\omega_q^{(j-\beta)k'} = 1$. If $a_{p+ik'}(0) \neq 0$ for an i, this shows that the normalization of Γ is factored through the covering $(x,y) \to (x^{k'}, y^{k'})$. Therefore we have $k' = 1$ and hence $\omega_q^{j-\beta} = 1$. This completes the proof of the claim. The claim says that the action μ of the subgroup $\mathbb{Z}_q \subset L(G)$ is effective. The argument goes the same way for p by using f^{-1} and transposing the roles of p and q. Therefore the action μ is effective. If p, q satisfy Condition (4), then $L(G) = \mathbb{Z}_n$, $5 \leq n$

and Theorem 4.1 (3) applies to say that $\tilde{h}_u$, $\tilde{h}'_u$ are $\pm$ holomorphic with respect to z and u. This completes the proof of (4).

Proof of (5). For simplicity we prove only for the case where $r = 0$ and Γ_0 is a union of an irreducible curve Γ of type (2,2) and a graph of a non singular function g. Let $\tilde{\Gamma} = \{y=f(x)\}$ be an irreducible component of $P^{-1}(\Gamma)$, $P(x,y) = (x^2,y^2)$ and let g' be a function such that $g'(z)^2 = g(z^2)$. By the definition in the begining of this section, the group $G(\Gamma_0)$ is generated by the diffeomorphisms $\pm z$, $f(-f^{-1})$ and $g'(f^{-1})$. Assume that G is solvable. By (1), $G(\Gamma)$ is commutative if and only if $G^0 = \{z\}$ and then Γ is not irreducible as (2,2) are not coprime. So assume that $G^0 \neq \{z\}$ consists of k-flat diffeomorphisms. The linear term b of $g'(f^{-1})$ acts on $a \in \Lambda(G(\Gamma_0)) \subset \mathbf{C}$ to give $b^k a$. So if b is neither ω_{nk}, $k = 1,2,3,4$ nor $c\ \omega_{2k}$, $c \in \mathbf{R}$, then $\Lambda(G(\Gamma_0)) \subset \mathbf{C}$ is dense. Thus Theorem 4.1 (2) applies and the rigidity holds.

Proof of (6). Let Γ_u be a union of graphs of functions $f_{ui}: \mathbf{C},0 \to \mathbf{C},0$, $i = 1,2,3$. Then the group $G(\Gamma_u)$ is generated by $f_{u2}f_{u1}^{-1}$ and $f_{u3}f_{u1}^{-1}$. If the functions are generic, the group is non solvable and then the rigidity holds.

Proof of (7). This is the same as Theorem 5.2 (3).

Proof of (8). Assume that $r = 0$, Γ is irreducible of type (3.3) for simplicity and the group $G(\Gamma)$ is non commutative but solvable. We have seen in the proof of (2) that the linear term $L(G) = \mathbf{Z}_3$ acts on $\Lambda(G)$ effectively. If $\Lambda(G) \subset \mathbf{C}$ is dense, the rigidity holds by Theorem 4.1 (2). If $\Lambda(G)$ is not dense, then it is a triangular lattice and ker L embeds to $\exp t\chi$, $\chi(z) = z^{k+1}\ \partial/\partial z$ by Theorem 2.1 (1). We may assume that G/G^0 is generated by $\omega_4 z$ by Lemma 1.4. If Γ, Γ' are topologically equivalent, the same condition holds also for Γ' and hence $G(\Gamma)$, $G(\Gamma')$ are holomorphically conjugate. The case where Γ is not irreducible needs a detailed argument using explicit forms of other components. The proof will appear elsewhere.

The author does not know for which irreducible Γ the group is solvable even in the case Γ is of type (2,2).

Corollary 5.2. *Let* Γ_u, Γ'_u *be irreducible of type* $(p,q) \neq (2,2)$ *and topologically trivial as families of curves. Then* Γ_u, Γ'_u *are* $\pm$ *holomorphically equivalent if and only if topologically equivalent*.

Theorem 5.3 *Let* Γ_u, Γ'_u, $u \in \mathbb{C}^r$ *be diagrams of irreducible correspondences* Γ_{ui}, $\Gamma'_{ui} \subset \mathbb{C} \times \mathbb{C}$, $i = 1,\dots,k$ *of types* (p_i,q_i) *topologically trivial as families of curves*.

$$\Gamma_u: \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\Gamma_{u1}} & \\ \mathbb{C} & \xrightarrow{\Gamma_{u2}} & \mathbb{C} \\ \vdots & & \\ \mathbb{C} & \xrightarrow{\Gamma_{uk}} & \end{array}$$

(1) $G(\Gamma_u)$ *is commutative if and only if* Γ_u *is equivalent to a diagram consisting of* $\Gamma_{ui} = \{x^{q_i} = y^{p_i}\}$, $i = 1,\dots,k$.

Assume that Γ_u *and* Γ'_u *are topologically equivalent. If one of the following conditions holds, then the equivalence is* $\pm$ *holomorphic with respect to* z *and* u.

(2) $G(\Gamma_0)$ is non-solvable

(3) *The L.C.M. of* p_i, q_i *for* i *for which* $G(\Gamma_i)$ *is non-commutative is equal of greater than* 5.

(4) *A* Γ_{0i} *is of type either* (2,4), (4,2) *or* (4,4) *for which* $G(\Gamma_{0i})$ *is non-commutative*.

Proof. We assume r = 0 and prove only for Case (3). The other cases follows from Theorem 5.1. Assume that $G(\Gamma_i)$ are non-commutative and $G(\Gamma)$ is solvable but non-commutative. The group $G(\Gamma)$ is generated by the lifts the action of $G(\Gamma_i)$ to the covering $\tilde{\Gamma}$. The projection of $\tilde{\Gamma}$ to $\tilde{\Gamma}_i$ has L/q_i sheets and the monodromy of order L/q_i generates the lift $\tilde{G}(\Gamma_i)$ with the pull-backs of elements of $G(\Gamma_i)$ with the linear term z. The subgroup $\tilde{G}(\Gamma_i)^0 \subset \tilde{G}(\Gamma_i)$ is generated by these pull-backs. In the proof of Theorem 5.1 (4), we have proven that the action of $L(G(\Gamma_i))$ on $\Lambda(G(\Gamma_i))$ is effective. Therefore the diffeomorphisms in $\tilde{G}(\Gamma_i)$ with non trivial linear terms act on the subgroup $G(\Gamma)^0$ effectively. The linear term $L(G(\Gamma))$ is generated by those of the lifts. Thus its action on $\Lambda(G(\Gamma))$ is effective. Clearly $L(G(\Gamma)) = \mathbf{Z}_n$, n is the L.C.M. of p_i, q_i, $i = 1,\ldots,k$. The proof then follows that of Theorem 5.1 (4).

6 Separatrix of groups of diffeomorphisms

For solvable group G of holomorphic diffeomorphisms of $\mathbf{C},0$ the orbit structure is less complicated as being classified in Theorem 2.1 except for the case where the linear terms of the elements of G are Brjuno numbers. Assume that the diffeomorphisms f generating G are defined on neighbourhoods U_f of the origin. The orbit of $z \in \mathbf{C}$ under G is the set of images W(z) by words W of the generators defined at z. The *separatrix* $\Sigma(G)$

of G is a finite union of disjoint Jordan arcs C_i: $I,0 \to \mathbb{C},0$ real analytic off the origin invariant under G such that any orbit of G is dense or empty in each chamber of $\mathbb{C}-\Sigma$ at the origin and any sub-union of the arcs does not possesses the property. The existence was proved by Shcherbakov [15] in a weak sense. The following theorem and the proof show the existence and the uniqueness of the separatrix.

Theorem 6.1 *For non-solvable* G_u, *the separatrix* $\Sigma(G_u)$ *exists for any domain of definition* U_f, *of which the germ at the origin is independent of the choice of sufficiently small* U_f.

The separatrix $\Sigma(G)$ is a union of jordan arcs which have tangent lines at the origin. The linear term $L(G) \subset \mathbb{C}$ acts on the tangent cone $T_0\Sigma(G)$.

Proof. Let χ, μ be the vector fields on the i-sheeted covering $\mathbb{C}_i$ constructed in Section 3. The χ defines the germ of vector field χ' on $\mathbb{C},0$ defined on the angular domain D_χ of those z for which $f^n(z) \to 0$ as $n \to \infty$. By the form $\chi(\tilde{z}) = dF^{-\infty}(-bi(-ai)^{j/i}$ $\partial/\partial\tilde{z})$ and the estimate $|dF^{-\infty}| \leq K'|\tilde{z}|^{-1/i}$ in the proof of Lemma 3.3, we obtain

$$\chi'(z) = (-b\ (-ai)^{j/i}\ z^{i+1} + O(Z^{i+2}))\ \partial/\partial z$$

Similarly we see that μ defines the vector field μ' defined on the angular domain D_μ of the form $\mu'(z) = (-c\ (-ai)^{j/i}\ z^{i+1} + O'(z^{i+2}))\ \partial/\partial z$.

By the argument in the proof of Proposition 3.6 the trajectories of χ', μ' passing through z are arbitrary well approximated by the orbits of type $f^{-n}g^m f^n(z)$, $g^{-n}[f,g]^m g^n(z)$ with sufficiently large m, n. So if $\chi'(z)$, $\mu'(z)$ are linearly independent over $\mathbf{R}$, then any orbit of G is dense or empty in a neighbourhood of z. The set $\Sigma(f,g)$ of those z where χ', μ' are independent is a smooth curve defined by $(\chi'/\mu')^{-1}(\mathbf{R})$, which is real analytic off the origin. By the above forms of χ', μ' we see that $\Sigma(f,g)$ is a finite union of disjoint jordan arcs in $D_\chi \cap D_\mu$ which have unique tangent directions at the origin. For any small $z \in \mathbf{C}$ either forward or backward orbit by f, g tend to the origin as n tends to infinity. So a neighbourhood of the origin is covered with the intersections of the angular domains $D_{\chi_{\pm 1}} \cap D_{\mu_{\pm 1}}$, where $\chi_{\pm 1}$, $\mu_{\pm 1}$ are respectively defined similarly to χ, μ with the pairs $(f^{\pm 1},g)$, $(f,g^{\pm 1})$. Define the subsets $\Sigma(f^{\pm 1},g^{\pm 1})$ of $D_{\chi_{\pm 1}} \cap D_{\mu_{\pm 1}}$ similarly to the case of $\Sigma(f,g)$. Let $\Sigma'(G)$ be the set of those $z \in \Sigma(f^i,g^j)$, $i, j = \pm 1$ where all other $\Sigma(f^{i'},g^{j'})$ coincide with $\Sigma(f^i,g^j)$. By the real analyticity of the components of $\Sigma(f^{\pm 1},g^{\pm 1})$, we see that $\Sigma'(G)$ is a finite union of the components of $\Sigma(f^{\pm 1},g^{\pm 1})$, and the local density of the orbits off $\Sigma(f^{\pm 1},g^{\pm 1})$ implies that the orbits are dense or empty in each chamber of $\mathbf{C} - \Sigma'(G)$ at the origin. Secondly define $\Sigma''(G)$ by deleting arcs L of $\Sigma'(G)$ such that the orbit G(L) is not contained in $\Sigma'(G)$ at the origin. Finally define the separatrix $\Sigma(G)$ by deleting arcs L of $\Sigma''(G)$ facing two chambers on both of which an orbit is dense at the origin. Then the resulting set $\Sigma(G)$ is invariant under G and any orbit is dense or empty in each chamber of $\mathbf{C} - \Sigma(G)$ at the origin.

The set $\Sigma''(G)$ is defined by the local properties of the orbits nearby the origin, so the germ is independent of the choice of U_f. We consider the final step of the construction of the separatrix. Assume that for any U_f there exists an orbit dense at the origin in two chambers with a common face L, in other words the chambers are joined by an arbitrary small orbit nearby the origin. Then L is deleted in constructing $\Sigma(G)$ from $\Sigma''(G)$ for any U_f. Thus the germ of the separatrix is independent of the choice of small U_f. The uniqueness follows from the definition.

Problem 1. Study the bifurcation of the separatrices.

Finally we propose a problem on global rigidity of algebraic correspondences. Let $\Gamma_i \subset C_i \times C_i'$, $i = 1,2$ be algebraic curves in products of Riemannian surfaces. We say Γ_1, Γ_2 are *topologically equivalent* if there exist homeomorphisms $h\colon C_1 \to C_2$ and $h'\colon C_1' \to C_2'$ such that $h \times h'(\Gamma_1) = \Gamma_2$. If the germ of Γ_1 at $x_0 \times y_0$ satisfies the condition in Theorem 5.1, h and h' are $\pm$ holomorphic at x_0, y_0 respectively. The natural question is whether the holomorphicity extends to the total space. The equivalence relations of $x \in C$, $y \in C'$ and $(x,y) \in \Gamma$ associated with Γ are generated by the relations

$$x \sim x' \in C \iff x, x' \in \Gamma \cap C \times y \quad \text{for a } y \in C',$$

$$y \sim y' \in C' \iff y, y' \in \Gamma \cap x \times C' \quad \text{for an } x \in C.$$

$$(x,y) \sim (x',y') \in \Gamma \iff x \sim x' \text{ or } y \sim y'.$$

We seek the geometry of the equivalence classes (orbits) $O_C(x) \subset C$, $O_{C'}(y) \subset C'$ and $O_\Gamma(x,y) \subset \Gamma$. The fundamental problems for the rigidity are

Problem 2. Determine the condition that there exists global separatrices $\Sigma_C(\Gamma) \subset C$, $\Sigma_{C'}(\Gamma) \subset C'$ and $\Sigma_\Gamma \subset \Gamma$, on each component of which the orbits are dense or empty.

Problem 3. Consider the orbits in Γ of multiple points $(x,y) \in \Gamma$, where the Γ is not of type (1,1).

More in general the topological structure of portraits of orbits of correspondences seems to possess complete analytic information. For the case where C, C' are Riemann spheres $\mathbf{P}$, the first and the second projections of Γ define two pencils of divisors on the normalization $\tilde{\Gamma}$. Then the problem is closely related to the geometry of divisors in linear systems of complex manifolds. From this point of view we finally refer to the following theorem.

Theorem 6.2 (The rigidity theorem of divisors [15]) *Let* C, C' *be Riemannian surfaces of genus* ≥ 2. *Assume that there exists a homeomorphism* $h : C \to C'$ *such that* h(D) *is a canonical divisor of* C' *for any canonical divisor* D *of* C. *Then* h *is holomorphic or anti-holomorphic diffeomorphism respectively whether* h *is orientation preserving or not*.

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Morse Inequalities for R-constructible Sheaves

——-a joint work with P. Schapira (Univ. Paris Nord) *

by

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Contents

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This note aims at giving a generalization of classical Morse inequalities for Betti numbers of compact manifolds (cf. [M]). In this paper, we work with $\mathbf{R}$-constructible sheaves instead and encounter the tight relation between Morse theory and Microlocal Analysis of Sheaves. See Witten[W] and Hellfer-Sjöstrand[H-Sj1,2] for another approach to Morse inequalities via microlocal analysis and also Goresky-MacPherson[G-McP] who introduced the "stratified Morse theory". This paper may be considered as a small variation on Kashiwara's index theorem [K], and in fact our proof is a slight modification of his.

1. Classical Morse inequalities

First of all, we recall what the classical Morse inequalities are.

Let X be a compact C^∞ manifold, and $\phi : X \longrightarrow \mathbf{R}$ a C^∞ function with the properties:

$$\{x \in X;\ d\phi(x) = 0\} = \{x_1, \ldots, x_N\}, \tag{1.1}$$

$$\begin{cases} x_j \text{ is a non-degenerate critical point of } \phi \text{ for any } j, \text{ i.e.} \\ Hess(\phi)(x_j) \text{ is non-degenerate.} \end{cases} \tag{1.2}$$

We set

$$n_k = \#\{x_j;\ \text{ the number of negative eigenvalues of } Hess(\phi)(x_j) = k\},$$

and

$$n_k^* = (-)^k \sum_{k' \le k} (-)^{k'} n_{k'}.$$

Moreover we set

$$b_k(X) = dim\ H^k(X; \mathbf{R}),$$

and

$$b_k^* = (-)^k \sum_{k' \le k} (-)^{k'} b_{k'}(X).$$

Then we have

Theorem 1.1 (cf. [M].) *We have for any* $k \in \mathbf{Z}$,

$$b_k^*(X) \leq n_k^*.$$

2. Preliminaries

Let X be a C^∞ manifold, and $\pi : T^*X \longrightarrow X$ its cotangent bundle. Let k be a commutative field. We denote by $D^b(X)$ the derived category with bounded cohomologies of sheaves of k-vector spaces on X.

2.1. *Micro-supports of sheaves.* Let $F \in ob(D^b(X))$, and $p = (x_0; \xi \cdot dx) \in T^*X$. Then $p \notin SS(F)$ if and only if

$$\begin{cases} \text{there exists a neighborhood } U \text{ of } p \text{ in } T^*X \text{ such that} \\ \text{for any } x_1 \in X \text{ and for any } C^\infty \text{ function } f \text{ with } f(x_1) = 0,\ df(x_1) \in U, \\ \text{we have} \\ \qquad \mathbf{R}\Gamma_{\{x;\ f(x) \geq 0\}}(F)_{x_1} = 0. \end{cases}$$

$SS(F)$ is called the micro-supporrt of F, and is a closed $\mathbf{R}_+$-conic set in T^*X. Refer to [K-S] for details.

2.2. $\mathbf{R}$*-constructible sheaves.* Now let X be a real analytic manifold. Then a sheaf F on X is called $\mathbf{R}$-constructible if there exists a subanalytic stratification

$$X = \sqcup_\alpha X_\alpha$$

such that

$$F\big|_{X_\alpha} \text{ is locally constant.}$$

Moreover $D^b_{\mathbf{R}-c}(X)$ denotes the sub-category of $D^b(X)$ consisting of objects with $\mathbf{R}$-constructible cohomologies. (cf. [K-S].) We remak that if $F \in ob(D^b_{\mathbf{R}-c}(X))$, $SS(F)$

is a Lagrangean subanalytic subset in T^*X.

2.3. *Pure sheaves.* Let X be a C^∞ manifold, and $p \in T^*X$. Then we set

$$\lambda_0(p) = T_p\pi^{-1}(p).$$

For a Lagrangean submanifold Λ passing through p, we put

$$\lambda_\Lambda = T_p\Lambda.$$

Now let ϕ be a real C^2 function on X. Then we set

$$\Lambda_\phi = \{(x; d\phi(x));\ x \in X\}.$$

Moreover for $p \in \Lambda_\phi$, $\lambda_\phi(p)$ denotes the tangent space to Λ_ϕ at p:

$$\lambda_\phi(p) = T_p\Lambda_\phi.$$

We will assume the condition

$$\text{(2.1)} \qquad \Lambda_\phi \text{ and } \Lambda \text{ intersect transversally at } p.$$

Under the above notation, we give

Definition 2.1. Let $F \in ob(D^b(X))$, and assume $SS(F) \subset \Lambda$ in a neighborhood of p. If for any real C^2 function ϕ satisfying the condition (2.1) we have

$$\mathbf{R}\Gamma_{\{\phi \geq \phi(\pi(p))\}}(F)_{\pi(p)} = k^m[\delta]$$

with

$$\delta = d - \frac{1}{2} dim\ X - \frac{1}{2}\tau(\lambda_0(p), \lambda_\Lambda(p), \lambda_\phi(p)),$$

F is called pure with shift d and multiplicity m. Here $\tau(\cdot,\cdot,\cdot)$ denotes the Maslov index. (Refer to [K-S].)

Consider in particular the case that Λ satisfies

$$\Lambda = T^*_Y X \quad \text{in a neighborhood of } p.$$

Let $x = \pi(p)$, and let $s^{\pm}(x)$ denote the number of positive or negative eigenvalues of $Hess(\phi\,|_Y)(x)$. Then we have

$$\tau(\lambda_0(p), \lambda_\Lambda(p), \lambda_\phi(p)) = s^-(x) - s^+(x). \tag{2.2}$$

3. Statement of the Main Theorem

Let k be a commutative field of characteristic 0. We denote by $Mod^f(k)$ the category of finite dimensional k-vector spaces and by $D^b(Mod^f(k))$ its derived category with bounded cohomologies.

Let $b = \{b_l\}_{l\in\mathbf{Z}}$ be a sequence of integers with $b_l = 0$ for $|l| >> 0$. We define

$$b^*_l \;=\; (-1)^l \sum_{j\le l} (-1)^j b_j, \qquad b^*_\infty \;=\; \sum_j (-1)^j b_j. \tag{3.1}$$

If $V \in ob(D^b(Mod^f(k)))$, we set

$$b_l(V) \;=\; dim\ H^l(V)$$

(and we define $b^*_l(V)$ and $\chi(V) = b^*_\infty(V)$ as in (3.1).)

Now let $F \in ob(D^b(X))$, and let $\phi :\ X \longrightarrow \mathbf{R}$ be a real valued C^2 function on X. We set

$$\Lambda = SS(F),$$

$$\Lambda_\phi \;=\; \{(x, d\phi(x)) \in T^*X;\ x \in X\ \}.$$

We shall assume:

$$\phi^{-1}(]-\infty, t]) \cap supp(F) \text{ is compact for any } t \in \mathbf{R}, \tag{H.1}$$

(in particular ϕ is proper on $supp(F)$),

$$\Lambda_\phi\ \cap\ \Lambda\ \text{ is finite,} \tag{H.2}$$

and setting $\Lambda_\phi \cap \Lambda = \{p_1, \cdots, p_N\}$, $x_j = \pi(p_j)$:

$$(H.3) \qquad V_j \overset{\text{def}}{=} \left(\mathbf{R}\Gamma_{\{x;\ \phi(x)\geq\phi(x_j)\ \}}(F)\right)_{x_j} \text{ belongs to } D^b(Mod^f(k)).$$

Set

$$n_l = \sum_j dim\ H^l(V_j).$$

Then we have

Theorem 3.1 (generalized Morse inequalities).

Assume (H.1), (H.2) and (H.3). Then:

$$(i) \qquad \mathbf{R}\Gamma(X; F) \in ob(D^b(Mod^f(k))),$$

(ii) setting $b_l(X, F) = b_l(\mathbf{R}\Gamma(X; F))$, we have the inequalities

$$b_l^*(X, F) \leq n_l^*.$$

Remark 3.2. The conclusion (i) is already obtained in [K]. Moreover since $b_l(X, F) = n_l = 0$ for $l >> 0$, we have

$$\chi(X; F) = n_\infty^*.$$

Here $\chi(X; F)$ is the Euler-Poincaré index of F on X. This is an obious version of Kashiwara's index theorem (cf. [K]).

4. Proof of the Main Theorem

In order to prove the theorem, we note

Lemma 4.1. *Consider a distinguished triangle in* $D^b(Mod^f(k))$

$$V' \longrightarrow V \longrightarrow V'' \xrightarrow{+1},$$

Then we have for any $l \in \mathbf{Z}$

$$b_l^*(V) \;\leq\; b_l^*(V') + \; b_l^*(V'').$$

(proof) We may assume that V, V' and V'' are concentrated in degree ≥ 0. Then we have a long exact sequence

$$0 \longrightarrow H^0(V') \longrightarrow H^0(V) \longrightarrow \ldots \longrightarrow H^l(V') \longrightarrow H^l(V) \longrightarrow B^l(V'') \longrightarrow 0,$$

where

$$B^l(V'') \;=\; Im(H^l(V) \longrightarrow H^l(V'')).$$

Then setting

$$\tilde{b}_l(V'') \;=\; dim\; B^l(V'')\; (j = l)$$

and

$$\tilde{b}_j(V'') \;= b_j(V'')\; (j < l),$$

we get:

$$b_l^*(V) \;=\; b_l^*(V') \; + (-)^l \sum_{l' \leq l} (-)^{l'} \tilde{b}_{l'}(V'').$$

Since $\tilde{b}_l(V'') \; \leq dim\; H^l(V'')$, the proof follows. *(Q.E.D.)*

[proof of Theorem 3.1.] We shall reduce the problem to the 1-dimensional case. To this purpose, we put

$$G \;=\; \mathbf{R}\phi_* F$$

and write

$$\phi(\{x_1,\dots,x_N\}) \;=\; \{t_1,\dots,t_L\}$$

with $t_j < t_{j+1}$. We set

$$\Lambda_t \;=\; \{(t;dt) \in T^*\mathbf{R};\ t \in \mathbf{R}\ \}.$$

First we remark that the hypothesis (H.1) is trivially satisfied by $(\mathbf{R}, t, G)$:

$(H'.1)$ $\qquad]-\infty, t] \cap supp(G)$ is compact for any $t \in \mathbf{R}$.

Next, applying Proposition 4.1 of [K-S] we get

$(H'.2)$ $$SS(G)\ \cap\ \Lambda_t\ \subset\ \{(t_i;dt) \in T^*\mathbf{R};\ i = 1,\dots,L\}.$$

Finally we have

$(H'.3)$ $$\left(\mathbf{R}\Gamma_{\{t\geq t_i\}}G\right)_{t_i} \;\simeq\; \bigoplus_{\phi(x_j)=t_i} V_j.$$

In fact

$$\begin{aligned}\left(\mathbf{R}\Gamma_{\{t\geq t_i\}}G\right)_{t_i} &\simeq \left(\mathbf{R}\Gamma_{\{x;\ \phi(x)\geq t_i\}}(F)\right)\big|_{\phi^{-1}(t_i)} \\ &\simeq \bigoplus_{\phi(x_j)=t_i} \left(\mathbf{R}\Gamma_{\{x;\ \phi(x)\geq t_i\}}F\right)_{x_j}\end{aligned}$$

by the definition of the micro-support. Thus the triple $(\mathbf{R}, t, G)$ satisfies the same hypotheses as (X, ϕ, F). Since

$$\mathbf{R}\Gamma(X;F) \;=\; \mathbf{R}\Gamma(\mathbf{R};G),$$

we have

$$b_l(X,F) \;=\; b_l(\mathbf{R},G),$$

and it is enough to prove the theorem for $X = \mathbf{R},\ \phi(t) = t$.

We set $X = \mathbf{R}$. Put $t_0 = -\infty,\ t_{L+1} = +\infty$, and define

$$I_t =]-\infty, t[,\ Z_t =]-\infty, t],\ I_j = I_{t_j},\ Z_j = Z_{t_j}.$$

Introduce:

$$b_l^*(Z_j, F) = b_l^*(\mathbf{R}\Gamma(Z_j; F)),$$

$$b_l^*(I_j, F) = b_l^*(\mathbf{R}\Gamma(I_j; F)).$$

Then by Theorem 1.4.3 of [K-S], we have the isomorphism

$$H^k(I_{j+1}; F) \simeq H^k(I_t; F) \quad (t_j < t \leq t_{j+1}).$$

By taking the inductive limit of the right hand side, we derive

$$H^k(I_{j+1}; F) \simeq H^k(Z_j; F). \tag{4.1}$$

Consider the distinguished triangle

$$\left(\mathbf{R}\Gamma_{\{t \geq t_j\}}(F)\right)_{t_j} \longrightarrow \mathbf{R}\Gamma(Z_j; F) \longrightarrow \mathbf{R}\Gamma(I_j; F) \xrightarrow{+1} . \tag{4.2}$$

Since $\mathbf{R}\Gamma(I_1; F) = 0$, we find by induction from (4.1) that both $\mathbf{R}\Gamma(Z_j; F)$ and $\mathbf{R}\Gamma(I_j; F)$ belong to $D^b(Mod^f(k))$.

Moreover (4.1) gives

$$dim\ H^k(X; F) = \sum_{1 \leq j \leq L} \{dim\ H^k(Z_j; F) - dim\ H^k(I_j; F)\}.$$

Hence

$$b_l^*(X, F) = \sum_{1 \leq j \leq L} \{b_l^*(Z_j, F) - b_l^*(I_j, F)\}.$$

On the other hand, we get by Lemma 4.1 from (4.2)

$$b_l^*(Z_j, F) - b_l^*(I_j, F) \leq b_l^*(\left(\mathbf{R}\Gamma_{\{t \geq t_j\}}(F)\right)_{t_j}).$$

Hence we have

$$b_l^*(X, F) \leq \sum_{1 \leq j \leq L} b_l^*((\mathbf{R}\Gamma_{\{t \geq t_j\}}(F)_{t_j})) = n_l^*.$$

This is the desired result. *(Q.E.D.)*

5. Application to pure sheaves

Let X be a real analytic manifold, and let $F \in ob(D^b_{\mathbf{R}-c}(X))$. Then $\Lambda = SS(F)$ is a Lagrangean subanalytic subset of T^*X. Take a real valued C^2 function ϕ on X and suppose

(5.1) $$\phi^{-1}(]-\infty, t]) \cap supp(F) \text{ is compact for any } t \in \mathbf{R},$$

(5.2) $$\Lambda_\phi \cap \Lambda = \Lambda_\phi \cap \Lambda_{reg} = \{p_1, \cdots, p_N\},$$

(5.3) $$\Lambda_\phi \text{ and } \Lambda_{reg} \text{ intersect transversally at each point } p_i\ ,$$

(5.4)

F is pure at each p_i with multiplicity m_i and shift d_i along Λ in the sense of [K-S].

Recall that (5.4) is equivalent to

(5.5) $$\left(\mathbf{R}\Gamma_{\{\phi(x)\geq\phi(x_i)\}}(F)\right)_{x_i} = k^{m_i}[\delta^i]$$

where $x_i = \pi(p_i)$, and

(5.6) $$\delta^i = d_i - \frac{1}{2}dim\ X - \frac{1}{2}\tau(\lambda_0(p_i), \lambda_\Lambda(p_i), \lambda_\phi(p_i)).$$

See Chapter 7 of [K-S] for the definition of Maslov index $\tau(\cdot,\cdot,\cdot)$. Under the above conditions, we get by Theorem 1.1

(5.7) $$\mathbf{R}\Gamma(X; F) \in ob(D^b(Mod^f(k))).$$

We set

(5.8) $$n_l = \sum_{\delta^i=-l} m_i, \qquad n_l^* = (-)^l \sum_{j\leq l}(-)^j n_j.$$

Then we have, by applying Theorem 1.1:

Theorem 5.1. *For any $l \in \mathbf{Z}$, we have the inequality*

$$b_l^*(X,F) \leq n_l^*. \tag{5.9}$$

Remark 5.2. Assume moreover:

$$\Lambda = T_{S_i}^* X \quad \text{in a neighborhood of } p_i, \tag{5.10}$$

where S_i is a real analytic submanifold of X. By (5.3), $\phi \mid_{S_i}$ is a Morse function at $x_i = \pi(p_i)$. Let $s^{\pm}(x_i)$ be the number of positive or negative eigenvalues of the Hessian of $\phi \mid_{S_i}$ at x_i. Then under the notation (5.6), we have

$$\delta^i = d_i - \frac{1}{2} dim\ X + \frac{1}{2}(s^+(x_i) - s^-(x_i)). \tag{5.11}$$

Remak moreover that if X is a complex manifold, F has $\mathbf{C}$-constructible cohomologies, and F is perverse, then we have

$$d_i = 0 \quad \text{for all } i.$$

Hence in this situation, we can deduce the Morse inequalities from the multiplicity of F at generic points of Λ.

6. An example of application to perverse sheaves

Let X be $\mathbf{C}^N$ $(N > 2)$ with coordinates $z = (z_1, \ldots, z_N)$ and set

$$S \;=\; \{z \in X;\ \sum_{1 \le j \le N} z_j^2 \;=\; 0\,\}.$$

We take $F \in ob(D^b_{\mathbf{C}-c}(X))$ (i.e. F has $\mathbf{C}$-constructible cohomologies) satisfying

$$\text{(6.1)} \qquad \Lambda \;=\; SS(F) \;\subset\; T^*_{S_{reg}}X \;\cup\; T^*_{\{0\}}X \;\cup\; T^*_X X,$$

$$\text{(6.2)} \qquad F \text{ is perverse.}$$

Recall that the perversity of F is equivalent to saying that F is pure with shift 0 at generic point of Λ (cf. Theorem 9.5.2 of [K-S]).

We put

$$\text{(6.3)} \qquad \Lambda_1 \;=\; T^*_{S_{reg}}X,\ \Lambda_N \;=\; T^*_{\{0\}}X,\ \Lambda_0 \;=\; T^*_X X$$

and assume that for any j

$$\text{(6.4)} \qquad F \text{ has multiplicity } m_j \text{ along } \Lambda_j.$$

We set

$$\text{(6.5)} \qquad \phi(z) \;=\; |z-a|^2 \text{ with } a = (1, 2\sqrt{-1}, 0, \ldots, 0).$$

Then we have

$$\Lambda_\phi \cap \Lambda_1 \;=\; \{p_{1,1} = (x_{1,1}; d\phi(x_{1,1})),\ p_{1,2} = (x_{1,2}; d\phi(x_{1,2}))\},$$

with

$$x_{1,1} \;=\; \left(\frac{-1}{2}, \frac{1}{2}\sqrt{-1}, 0, \ldots, 0\right) \text{ and } x_{1,2} \;=\; \left(\frac{3}{2}, \frac{3}{2}\sqrt{-1}, 0, \ldots, 0\right),$$

$$\Lambda_\phi \cap \Lambda_N \;=\; \{p_N = (0; d\phi(0))\},$$

and

$$\Lambda_\phi \cap \Lambda_0 = \{p_0 = (a;0)\}.$$

An easy calculation and the fornula (2.2) gives:

$$\tau(\lambda_0(p_{1,1}), \lambda_{\Lambda_1}(p_{1,1}), \lambda_\phi(p_{1,1})) = -2,$$

$$\tau(\lambda_0(p_{1,2}), \lambda_{\Lambda_1}(p_{1,2}), \lambda_\phi(p_{1,2})) = -2N+2,$$

$$\tau(\lambda_0(p_N), \lambda_{\Lambda_N}(p_N), \lambda_\phi(p_N)) = 0,$$

$$\tau(\lambda_0(p_0), \lambda_{\Lambda_0}(p_0), \lambda_\phi(p_0)) = -2N.$$

Hence we have

$$\begin{cases} n_0 = m_0,\ n_1 = m_1,\ n_{N-1} = m_1,\ n_N = m_N, \\ n_j = 0 \quad (j \notin \{0, 1, N-1, N\}). \end{cases}$$

This gives the inequalities:

$$b_l^*(X, F) \leq n^*.$$

Remark that in this example F is conic for the action of $\mathbf{R}^+$ on $\mathbf{C}^n$. Hence $\mathbf{R}\Gamma(X;F) \simeq F_0$, the stalk of F at 0.

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M-determinacy of Smooth Map-germs

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1. Results

The notions and notations are given in [3] or [2].

Theorem 1. Let $f: (N, x_0) \dashrightarrow (p, y_0)$ be an FST germ, $I \subset C_N$ an ideal.

$$I^r \theta_f \subset T A_k f$$

Then f is I^{2r+1}-A_k-determined.

Corollary 2. If $f: (N, x_0) \dashrightarrow (p, y_0)$ is an FST germ, and

$$m_N^\infty \theta_f \subset TA_k f$$

Then f is ∞ - A_k-determined.

Note. When $k=0$, this is the main result of Wilson [5].

Theorem 3. If f is M-A_k-determined (w.r.t. $\{y_1 \ldots y_p\}$), then $\quad M \subset TA_k f + m_N^{l+1} \quad \forall\ l \geq 0$

Theorem 4. Let $f: (N, x_0) \dashrightarrow (p, y_0)$ be an FST germ, $M \subset C_N^{\times p}$, a submodule. If there is a C_N - module K s.t.

$m_N^{k+1} \theta_N \subset K \subset m_N^k \theta_N$,

$$K \cdot M + \left(\bigcup_{j=1}^{p} M_j\right) \theta f \subset TA_k f$$

for some $s = 1$: $(i = 1, \ldots, s+1)$

a) C_N-modules $N_i \subset m_N^{k+1} \theta_N$ with $N_1 = m_N^{k+1} \theta_N$, $N_{s+1} = \{0\}$

b) $tf(K) + wf(m_p^k \theta_p) \supset A_j$ (as R-subspaces) with

$A_{1j} = tf(K) + wf(m_p^k \theta_p)$, $A_{s+1,j} = \{0\}$, $j=1, \ldots, P$

c) Let $T_{f,i} = tf(N_i) + \sum_{j=1}^{p} f^* y_j \cdot A_i j$, $R_i = N_i \cdot M + \sum_{j=1}^{p} M_j A_{ij}$

then

$$R_i \subset T_{f,i+1} + X_i\, T_{f\cdot i} + R_{i+1} + \sum_{l=1}^{i} X_l R_l \quad , \text{ where}$$

A_{ij} is an ($\mathbb{R} + X_i$) - module, X_i is a subring of C_N with $X_1 \subset \ldots \subset X_{s+1} = m_N$.

Then f is M-A_k-determined.

Note. When k=0, Theorems 1, 3 and 4 are the main results of [3].

2. Proofs

Lemma 1. The following are equivalent:

(1) f is of FST

(2) $\exists\, r,\ m_N^r \theta_f \subset TK_k f$

(3) $\exists\, s,\ m_N^s \subset J(f) + f^* m_P \cdot C_N$

Proof (1) $\Leftrightarrow$ (3), see [3] P.144

(1) $\Rightarrow$ (2), From [3] (1.20), $\exists$ s ,

$$m_N^s \theta_f \subset tf(m_N \theta_N) + f^* m_P \cdot \theta_f$$

$$\therefore \quad m_N^{s+k} \theta_f \subset tf(m_N^{k+1} \theta_N) + f^* m_P \cdot m_N^k \theta_f = TK_k f$$

(2) $\Rightarrow$ (1) $m_N^r \theta_f \subset TK_k f \subset TKf$, so f is of FST by [3] (1.20)

Lemma 2. Let f be an FST germ, and suppose

$$TK_k G \supset TK_k F \ , \qquad F = f \times 1_{(\mathbb{R},a)}$$

Then G is of FST.

Proof. f is of FST, then lemma 1 implies that there is an integer s, $m_N^s \theta_f \subset TK_k f$

Multiplying by $C_{N\times\mathbb{R}}$,

$$m_N^s\,\psi_F \subset TK_kF$$

$$m_N^s\psi_G = m_N^s\psi_F \subset TK_kG \subset TKG, \quad wG(\frac{\partial}{\partial t}) = (0,0,\ldots,0,1)$$

$$tG(\frac{\partial}{\partial t}) = (\frac{(y_1 \circ G)}{t}, \ \ldots, \ \frac{(y_p \circ G)}{t}, \ 1)$$

$$wG(\frac{\partial}{\partial t}) \in tG(\frac{\partial}{\partial t}) + \psi_G$$

Since $\theta_G = \psi_G + wG(\frac{\partial}{\partial t})\, C_{N\times\mathbb{R}}$, $\quad (m_N^s + t)\,\theta_G \subset TKG$,

$\therefore\ \exists\ r$, $m_{N\times\mathbb{R}}^r\,\theta_G \subset TKG$. Then G is of FST by lemma 1.

Proof of Th. 1. Let g be an I^{2r+1}-approximation to f.

$(w, r, t, \{y_1,\ldots,y_p\})$

$F = f\times 1_{(\mathbb{R},[0,1])}$, $G(x,t) = (g_t(x),t)$, $g_t(x)=(1-t)f(x)+tg(x)$

$\forall\ a \in [0,1]$, we denote F^a, G^a by F, G for convenience.

We have $J(f)\,C_{N\times\mathbb{R}} = J(F)$, $J(F)\,\psi_F \subset tF(\psi_{N\times\mathbb{R}})$. By [3] (1.22)

$$\{J(f)\,C_{N\times\mathbb{R}} + F^* C_{P\times\mathbb{R}}\}\ TA_kf = TA_kF$$

and

$$I^r\psi_F = I^r C_{N\times\mathbb{R}}\,\theta_f = I^r\{J(f)C_{N\times\mathbb{R}} + F^*C_{P\times\mathbb{R}}\cdot C_N\}\,\theta_f \subset TA_kF \qquad (1)$$

from [3] P.141,

$$TA_kG \subset TA_kF + I^{2r}\psi_F, \quad TA_kF \subset TA_kG + I^{2r}\psi_F \qquad (2)$$

$$\therefore\ TA_kG \subset TA_kF$$

By (1) $\quad I^{2r}\psi_F \subset I^r TA_kF \subset I^r TK_kF \qquad (3)$

Then $\quad TK_kF \subset TK_kG + I^{2r}\psi_F \subset TK_kG + m_N^{k+1}\,TK_kF \qquad (4)$

Set

$$E = \frac{TK_kF + TK_kG}{TK_kG}$$

It is a finitely-generated $C_{N\times\mathbb{R}}$-module. It follows from (4)

$$E = m_{N\times\mathbb{R}}\cdot E$$

By the Nakayama lemma, $E = 0$, and so

$$TK_k F \subset TK_k G$$

Then $I^{2r}\psi_F \subset I^r . TK_k G \subset G(m_N^{k+1}\psi_{N\times\mathbb{R}}) + G^* m_P . (I^r \psi_G)$

Substituting in (2)

$$TA_k F \subset TA_k G + G^* m_P(I^r\psi_G) \subset TA_k G + G^* m_P . TA_k F$$

Therefore

$$TA_k G \subset TA_k F \subset TA_k G + G^* m_P . TA_k F \tag{6}$$

Let $E_1 = TA_k F \;/\; TA_k G$, then:

(a) E_1 is a $G^* C_{P\times\mathbb{R}}$-module:

$\because$ $TA_k F$ is an $F^* C_{P\times\mathbb{R}}$- module, and $I^r C_{N\times\mathbb{R}} . TA_k F$

$\subset I^r C_{N\times\mathbb{R}} \psi_F \subset TA_k F$

by [3] (1.13), $(F.G)^* C_{(P\times\mathbb{R})\times(P\times\mathbb{R})} \subset F^* C_{P\times\mathbb{R}} + I^r C_{N\times\mathbb{R}}$,

$$(F.G)^* C_{(P\times\mathbb{R})\times(P\times\mathbb{R})} . TA_k F \subset TA_k F$$

It means that $TA_k F$ is an $(F,G)^* C_{(P\times\mathbb{R})\times(P\times\mathbb{R})}$-module and then is a $G^* C_{P\times\mathbb{R}}$-module, so E_1 also is.

(b) E_1 is finitely-generated:

By (5) and lemma 2, G is of FST. Hence (lemma 1) $\exists$ $s < \infty$,

$$m^s_{N\times\mathbb{R}} \subset J(G) + G^* m_{P\times\mathbb{R}} . C_{N\times\mathbb{R}}$$

Since $m_{N\times\mathbb{R}} = (m_N + toG) C_{N\times\mathbb{R}}$

$$m^{s+k+1}_{N\times\mathbb{R}} \subset m_N^{k+1} J(G) + G^* m_{P\times\mathbb{R}} . C_{N\times\mathbb{R}}$$

then $C_{N\times\mathbb{R}} / \left\{ m_N^{k+1} J(G) + G^* m_{P\times\mathbb{R}} . C_{N\times\mathbb{R}} \right\}$ is a finite-dimensional real space and $C_{N\times\mathbb{R}} / m_N^{k+1} J(G)$ is a finitely-generated $C_{N\times\mathbb{R}}$ - module.

By the Malgrange Preparation Theorem, $C_{N\times\mathbb{R}} / m_N^{k+1} J(G)$ is a finitely-generated $G^* C_{P\times\mathbb{R}}$-module. Let $h_1, \ldots, h_q \in C_{N\times\mathbb{R}}$ project to a spanning set. Let

$$E_2 = TA_kF \,/\, tG\,(m_N^{k+1}\,\psi_{N\times \mathbb{R}})$$

Because $tF\,(m_N^{k+1}\cdot J(G)\,\psi_{N\times\mathbb{R}}) \subset m_N^{k+1}\; J(G)\,\psi_F$

$$= m_N^{k+1}\; J(G)\,\psi_G \subset m_N^{k+1}\; tG\,(\psi_{N\times\mathbb{R}}),$$

it is easy to show that E_2 is a finitely-generated $(F,G)^*$ $C_{(P\times\mathbb{R})\times(P\times\mathbb{R})}$ -module. By means of the Malgrange Preparation Th., we can know that E_2 is a finitely -generates $G^*C_{P\times\mathbb{R}}$ -module, so is E_1 ($\because$ E_1 is the quotient module of E_2)

(c) From (2), we have $E_1 = G^* m_{P\times\mathbb{R}}\cdot E_1$. By the Nakayema Lemma and (a) - (b), $E_1 = 0$, so $TA_kF = TA_kG$, then $I^r\psi_G = I^r\,\psi_F \subset TA_kG$ By [3, 141] , $\partial G \in I^r\,\psi_G \subset TA_kG$

Recall that $G = G^a$, $a \in [0, 1]$. By [2] P.122, G is A_k-trivial. Hence $f = g_0$ is A_k-equivalent to $g=g_1$, and f is I^{2r+1} - A_k-determined.

The proofs of Th. 3 and 4 are omitted.

3. Example

Example 5. Let $(N, X_0) = (\mathbb{R}^2, 0)$, $(P, y_0) = (\mathbb{R}^3, 0)$, and let $f: (N, x_0) \text{ -- } (P, y_0)$ be the map germ given by

$$f(x,y) = (x, y^2, y^3).$$

Let $M = (\{y^2\}\cdot C_N, \{y^2\}\,m_N, \{y^3\}\,m_N)$. By [3] (3.7), f is M-A-determined. But f does not M-A_2-determined. In fact, if f is M-A_2-determined, then from Th.3, we have

$$M \subset TA_2f + m_N^3\,\theta_f \subset m_N^3\;\theta_f$$

But $(y^2, 0, 0) \in M$ and $(y^2,0,0) \notin m_N^3\,\theta_f$

This example shows that "M-A-det." may be not "M-A_k-det.". But Plessis [2] (2.8) shows that for finitely-determinacy, this is true, i.e. "f is finitely-A-det." iff "f is finitely-A_k-det. for all $k \gtrsim 0$". Then M-determinacy is different from finitely-determinacy.

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On the topology of singular sets of smooth mappings

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§1. Introduction

Given any smooth mapping $f: M^n \to N^p$, the structure, mainly the topology, of singular sets of the mapping f seems generally intractable. Even if we restrict ourselves to stable mappings, the similar difficulties follow. For this reason, we will investigate the behavior of the singular sets by considering the case which the smooth mappings have only simple singularities. Some of our main results are the following,

Theorem 3.1

For an $A_1(n,p)$-type $f: M^n \to R^p$ $(p>2)$, if the Euler number of M^n is odd, then the singular sets $S(f)$ must be unorientable.

If we consider specific dimensions of source and target manifolds, we can formulate the topological structure of $S(f)$.

Theorem 4.5

Let M^4 be a closed, oriented 4-dimensional manifold with vanishing first intgral homology group. For an $A_2(4,3)$-type $f: M^4 \to R^3$ such that $S(f)$ is connected, it holds,

$$\sigma(M^4) \equiv -S(f) \cdot S(f) \mod 4.$$

Here $\sigma(M^4)$ denotes the signature of M^4, which is the number of positive eigenvalues minus the negative eigenvalues when the cup product form is defined on the middle dimensional cohomology, and the dot is self intersection number of $S(f)$ in M^4.

Theorem 4.7

Let M^4 and f be as above and k' stand for the number of connected characteristic surfaces in $S(f)$.

i) $\# S^{odd}(f)$: even

$$(k'-2)\sigma(M^4) \equiv \widetilde{S(f)}\cdot\widetilde{S(f)} \mod 4$$

ii) $\# S^{odd}(f)$: odd

$$(k'-2)\sigma(M^4) \equiv \widetilde{S(f)}\cdot\widetilde{S(f)} + 2 \mod 4$$

The most part of this paper is devoted to detailed formulations and discussion of the statements mentioned above. All manifolds and mappings considered are smooth of class C^∞.

§2. Morin singularities and key results

Let M^n be a closed, oriented n-dimensional manifold and N^p be a p-dimensional manifold with $n \geq p$. If we can choose coordinates $(x_1, x_2, \ldots, x_n)$ centered at x of M and $(y_1, y_2, \ldots, y_p)$ centered at f(x) of N so that f has equations;

$$y_i = x_i \quad (1 \leqq i \leqq p-1)$$

$$y_p = x_p^{k+1} + \sum x_i x_p^{k-i} \pm x_{p+1}^2 \ldots \pm x_n^2,$$

thenapoint x of M^n is called a Morin singularity of type A_k of f. In other words, let $S^i(f)$ stand for the set of all points of M^n which the rank of $\ker(df_x)$ equals i, then $S^i(f)$ is a submanifold of M^n for an appropriate mapping f and we can define $S^{i,j}(f)$ as $S^j(f|S^i(f))$. Let I_r be the r-sequence $(i,1,\ldots,1)$, $i=\max\{1,n-p+1\}$ then we can define $S^{I_r}(f)$ as $S^1(f|S^{I_{r-1}}(f))$ inductively. Then a point of $S^{i,0}(f)$ or $S^{I_r}(f)$ respectively ia a Morin singularity of symbol $(i,0)$ or I_r. We call a mapping $f\colon M^n \to N^p$ an $A_k(n,p)$-type if f has no other singularities than Morin ones of type A_i $(1 \leq i \leq k)$. In particular for k=1 it is usually called a fold point, which is the simplest singularity of all, and for an A_2-type a cusp point. We denote the set of singular points of a mapping f by S(f). According to Morin [4], the following properties are given for an $A_k(n,p)$-type $f\colon M^n \to R^p$.

Lemma 2.1 (B.Morin [4])

Let $A_k(f)$ denote the set of A_k-type singular points of f and $\overline{A_k}(f)$

be the topological closure of $A_k(f)$.

1) $A_k(f)$ and $\overline{A_k}(f)$ are both p-k dimensional submanifolds of M^n.
2) $\overline{A_k}(f) = \bigcup_{i \geq k} A_i(f)$.
3) The restricted mappings $f|A_k(f) \to R^p$ are immersions.

These facts give several applications and play a fundamental role as we will see later. Moreover, the following results have been recently shown concerning the geometrical structure of $A_k(f)$.

Lemma 2.2 (Fukuda[1])

Let M^n be a compact manifold and $f: M^n \to R^p$ an $A_k(n,p)$-type, then we have the congruence involving the Euler number,

$$\chi(M^n) \equiv \Sigma \chi(\overline{A_k}(f)) \bmod 2.$$

For an $A_1(n,p)$-type, $w_{n-i+1}(M^n) \neq 0$ implies $w_{p-i}(S(f)) \neq 0$ for any i with $1 \leq i \leq p-1$, where $w_j(\ )$ denetes the j-th Stiefel-Whitney class.

This lemma may be of independent interest and besides, it suggests various applications. For example, it is easy to see that if S(f) is orientable for an $A_1(n,n)$-type then M^n must must be spin.

Remark 2.3

It is likely that we might be able to replace R^p p-dimensional Euclidean space by a parallelizable manifold in lemma 2.2. However the author does not know whether it is true or not.

Our goal in next section is based on the following two examples.

Example 2.4

Suppose the Euler number of M^n is odd, then there does not exist $A_1(n,2)$-type mappings, that is, such a mapping necessarily has cusp points. Because the singular set is disjoint union of circles in this case hence the Euler number is zero lemma 2.2 says that it is not be able to happen.

Example 2.5

Suppose the Euler number of M^n is odd for an $A_1(n,3)$-type $f:M^n \to R^p$ then $S(f)$ is unorientable. Note that by lemma 2.1 $S(f)$ is a two dimensional manifold. This is because the Euler number of every orientable surface is even by well known classification of closed 2-manifolds, after all we can only take surfaces with odd genus as $S(f)$ which are unorientable.

§3. Characteristic classes

As stated in the previous section, $S(f)$ of $A_1(n,p)$-type f is on the geometrically conspicuous location so that $S(f)$ is immersible in one dimension higher Euclidean space R^p. We will generalize the above example which states the nonorientability of $S(f)$, using the fundamental properties of characteristic classes.

Theorem 3.1

Let $f: M^n \to R^p$ be an $A_1(n,p)$-type with $p \geq 3$. If the Euler number of M^n is odd, then $S(f)$ must be unorientable.

Before the proof of Theorem 3.1, we need a lemma.

Lemma 3.2

Let M and f be as in theorem 3.1. Then we can caluculate the total Stiefel-Whitney class of $S(f)$ as follows,

$$W(S(f)) = 1 + \hat{a} + \hat{a}^2 + \ldots + \hat{a}^{p-1}, \text{ where } \hat{a} = w_1(S(f)).$$

proof) At first we consider the following diagram,

$$\begin{array}{ccc} f^*T(R^p) & \longrightarrow & T(R^p) \\ \downarrow & & \downarrow \\ S(f) & \xrightarrow[f|S(f)=\tilde{f}]{} & R^p \end{array}$$

where $T(R^p)$ denotes the tangent bundle of R^p and $f^*T(R^p)$ the pull back. As usual define $v(\tilde{f})$, the normal bundle of the immersion $\tilde{f}$ by the exactness;

$$0 \to T(S(f)) \to f^*T(R^p) \to v(\tilde{f}) \to 0.$$

Then it holds,

$$T(S(f)) \oplus v(\tilde{f}) = f^*T(R^p).$$

This bundle isomorphism gives the following identity of the total Stiefel-Whitney classes by the parallelizability of R^p.

$$w(S(f))\cdot w(v(\tilde{f})) = 1.$$

Since $v(f)$ is a line bundle, we can set $w(v(\tilde{f})) = 1+\hat{a}$, where $\hat{a}$ is an element of first cohomology group of $S(f)$ over the coefficient $Z/2$. Hence,

$$w(S(f)) = 1 + \hat{a} + \hat{a}^2 + \ldots + \hat{a}^{p-1},$$

where $\hat{a}^k$ means $\underbrace{\hat{a}\cup\ldots\cup\hat{a}}_{k\text{-times}}$, cup product and $\hat{a} \in H^1(S(f);Z/2)$.

We have an immediate corollary from this lemma.

Corollary 3.3

Let M^n be an oriented manifold and $f: M^n \to R^n$ an $A_1(n,n)$-type. If $S(f)$ is orientable, all the Stiefel-Whitney numbers of M^n vanish.

proof) Since $S(f)$ is orientable, $\hat{a} = 0$ and $w_j(S(f)) = 0$ $(1\leq j\leq p-1)$ from lemma 3.2. Then, using lemma 2.2, we have $w_k(M^n) = 0$ $(2\leq k\leq n)$. Hence the conclusion follows.

proof of theorem 3.1) Assume the normal bundle $v(\tilde{f})$ of the immersion $\tilde{f}$ is trivial, that is, $\hat{a}$ ia a zero element. Thus we have $w_{p-1}(S(f))=0$ for $p>2$. This implies $w_n(M^n)=0$ by lemma 2.2. Then the top dimensional Stiefel-Whitney class is equal to the $Z/2$-Euler class $\hat{e}(M^n)\in H^n(M^n;Z/2)$. (See [3]) Therefore we obtain

$$\begin{aligned}\chi(M^n) &\equiv \langle \hat{e}(M^n), [M^n]_2 \rangle \mod 2,\\ &\equiv \langle w_n(M^n), [M^n]_2 \rangle \mod 2,\\ &\equiv 0 \mod 2.\end{aligned}$$

This contradicts our assumption. Hence we have shown that $w_1(S(f))$ is a nonzero element. This completes a proof.

Remark 3.4

In case p=2 we see there does not exist an $A_1(n,2)$-type f with M^n having odd Euler number, using the same argument as in the proof of theorem 3.1. It is of specific interest to compare our results with Levine's work [2].

§4. Embedding phenomena of S(f)

One of the basic problem of geometry lies in the characterization of manifolds by means of algebraic invariants. Characterizing the structure of manifolds, we have known that topology of 4-manifolds is particularly facinating object. As stated in the introduction we will investigate an $A_2(4,3)$-type $f: M^4 \to R^3$ i.e. a mapping which admits only fold and cusp singularities. In this case S(f) is a 2-dimensional submanifold of M^4. Though S(f) is smoothly embedded in M^4, the realization may not be generally trivial. Then it arises the question how the realization of S(f) in M^4 is related to the mapping and topology of M^4. In the rarefied realm of differential topology and singularity theory, we mainly concern ourselves with relations between the signature of M^4 and the self intersection number of S(f) determined by the mapping.

Definition 4.1

Let M^4 be a closed, oriented 4-manifold and F^2 be a closed surface properly embedded in M^4, which is not necessarily orientable. F^2 is called a characteristic surface of M^4 if the intersection number mod 2 $F\cdot x$ is equal to the self intersection number $x\cdot x$ for any x of $H_2(M^4;Z/2)$; otherwise it is called an ordinary surface. It is equivalent to saying that the homology class $[F^2]\in H_2(M^4;Z/2)$ is dual to the 2-nd Stiefel-Whitney class $w_2(M^4)$.

Lemma 4.2

For an $A_2(4,3)$-type $f: M^4 \to R^3$ with S(f) being connected, S(f) is a characteristic surface of M^4.

This lemma is a special case of the following proposition proven by R.Thom. We use the following sign.

$$\Sigma_i = \{ j^1 g(x) \in J^1(M^n, R^p) ; \mathrm{rank}\, dg_x = i \}$$

where $J^1(M^n,R^p)$ is the jet space from M^n to R^p.

Lemma 4.3 (R.Thom [5])

Let $f:M^n \to R^p$ $(n \geq p)$ be a smooth mapping such that 1-jet extension $j^1 f:M^n \to J^1(M^n,R^p)$ is transeversal to Σ_i $(1 \leq i \leq p-1)$. Then $S(f)$ is dual to the n-p+1-st Stiefel-Whitney class $w_{n-p+1}(M^n) \in H^{n-p+1}(M^n;Z/2)$.

This lemma 4.2 enables us to evaluate the self intersection number of S(f), making use of geometrical structures in four dimensional topology, as we will see later.
To begin with, we recall the " generalized Whitney's congruence" indicated by Rochlin .

Proposition 4.4 (Rochlin [6])

Let M^4 be a closed, oriented 4-dimensional manifold with $H_1(M;Z)=0$ and F^2 a characteristic surface of M^4. Then we have,

$$\sigma(M^4) \equiv F^2 \cdot F^2 + 2\chi(F^2) \mod 4.$$

Cobining lemma 4.2 and proposition 4.4, we obtain our congruence formula in singularity theory under the assumption which S(f) is connected.

Theorem 4.5

With the notations above, it holds

$$\sigma(M^4) \equiv -S(f) \cdot S(f) \mod 4.$$

proof) In our case since $A_2(f)$ is disjoint union of circles, its Euler number vanishes. Thus we have $\sigma(M^4) \equiv \chi(M^4) \equiv \chi(S(f))$ mod2.

Then the conclusion follows.

We should essentially know the relation between the topology of M^4 and the number of connected components simultaneously, because in general S(f) has several connected components. To deal with the general case, we need to provide the following notations.

$\#S^{odd}(f)$: the number of connected, unorientable ordinary surfaces with odd genus in S(f).

$\widetilde{S_i(f)}$: a connected characteristic surface in S(f).

$\widetilde{S(f)}$: the set of characteristic surfaces in S(f).

We assume S(f) has k connected components.

<u>Lemma 4.6</u>

<u>For</u> <u>an</u> $A_2(4,3)$-type $f: M^4 \to R^3$,

<u>If</u> $\#S^{odd}(f)$ <u>is</u> <u>even</u>,

$$\sigma(M^4) \equiv \chi(S(f)) \bmod 2.$$

<u>If</u> $\#S^{odd}(f)$ <u>is</u> <u>odd</u>,

$$\sigma(M^4) \equiv \chi(S(f)) + 1 \mod 2.$$

proof) Let $k' \leq k$ and $k'' < k$ be natural numbers. We assume S(f) has k' connected characteristic surfaces and k" connected ordinary ones. S(f) has the following decomposition,

$$S(f) = \widetilde{S_1(f)} \cup \ldots \cup \widetilde{S_{k'}(f)} \cup S_1(f) \cup \ldots \cup S_{k''}(f).$$

Thus we have,

$$\chi(S(f)) \equiv \chi(\widetilde{S(f)}) + \varepsilon \mod 2,$$

where

$$\varepsilon = \begin{cases} 1 & \#S^{odd}(f) : odd \\ 0 & \#S^{odd}(f) : even \end{cases}$$

Note that $\sigma(M^4) \equiv \chi(S(f)) \bmod 2$ as in the proof of theorem 4.5. Hence we have the required results.

We are in a position to prove the main theorem in this section.

Theorem 4.7

For an A (4,3)-type f we have the following congruence formulas respectively in that case.

$\# S^{odd}(f)$: even

$$(k'-2)\,\sigma(M^4) \equiv \widetilde{S(f)}\cdot\widetilde{S(f)} \mod 4.$$

$\# S^{odd}(f)$: odd

$$(k'-2)\,\sigma(M^4) \equiv \widetilde{S(f)}\cdot\widetilde{S(f)} + 2 \mod 4.$$

proof) Using proposition 4.4, we have

$$\sigma(M^4) \equiv \widetilde{S_i(f)}\cdot\widetilde{S_i(f)} + 2\chi(\widetilde{S_i(f)}) \mod 4.$$

for $1 \leq i \leq k'$.

Applying lemma 4.6 to this congruences, the conclusions follow.

Remark 4.8

Our evaluation works in nothing but modulo four. Accordingly,it is preferable to work in modulo sixteen, extracting the essence of 4-dimensional topology.

From the different point of view, $A_2(4,2)$-type $f: M^4 \to R^2$ has been recently investigated and interesting observations have been given by Kobayashi [2].

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Holonomy of Foliations with Singularities

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INTRODUCTION

We mean by a generalized foliation, a foliation with singular leaves in the sense of P. Stefan [St] and P.Dazord [D], and in the present paper we establish a notion of a holonomy groupoid of this generalized foliation. In order to keep similarities to the case of regular foliations, we set a limitation on singular leaves, however our result is applicable to some foliations of Poisson structures and to some foliations which are not locally simple (cf. [E]). In many cases, we call a generalized foliation simply a foliation.

According to [D], along a leaf F of a foliation $\mathcal{D}$ on a C^∞-manifold M there is a unique germ Δ_F of transverse structure. A singular leaf F is called *tractable* if F has a saturated neighborhood N in M with following properties:

(i) N is isomorphic to a fibre bundle over F, $\pi_F : N \to F$ having a fibre V with a foliation Δ_V which is a representative of Δ_F.

(ii) The structural group of the bundle is the group of isomorphisms of Δ_V and the foliation of N determined by a local product of the one leaf foliation of F and Δ_V is the restriction $\mathcal{D}_N$ of $\mathcal{D}$ to N.

We will show that if each singular leaf of $\mathcal{D}$ is tractable, then a holonomy groupoid $G(\mathcal{D})$ of $\mathcal{D}$ is constructed in a similar way to to that of a regular foliation.

The part of $G(\mathcal{D})$ outside singular leaves is a (non-Hausdorff) C^∞-manifold by the usual theory of regular foliations (see, e.g., [Wi]), but $G(\mathcal{D})$ itself is not a manifold in general. We take examples of the foliations from those of symplectic leaves of Poisson structures and from some constructions of fibre bundles. In his construction of a singular foliation C^*-algebra, A. Sheu [Sh] uses the notion of holonomy of a locally simple foliation due to C. Ehresmann [E], but there are some foliations which are not locally simple. Our definition of holonomy can be applied to these.

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1. Generalized foliations

Here we review basic facts about foliations with singularities from P.Dazord's work [D]. Let M be a paracompact Hausdorff C^∞-manifold and T_xM the tangent space of M at a point $x \in M$. Let $C^\infty(M)$ be the algebra of real valued C^∞-functions on M.

A *distribution* D of tangent subspaces of M is a collection of subspaces:

$$D = \{D_x \subset T_xM | x \in M\}.$$

Let $\rho(x)$ denote the dimension of D_x and $C^\infty(D)$ denote the $C^\infty(M)$-module of C^∞-vector fields on M, the value of which belongs to D_x for each $x \in M$.

D is called a *C^∞-distribution*, if for each $x \in M$ one can find a finite number of elements of $C^\infty(D)$,

$$X_1, \ldots, X_k$$

such that D_x is generated by $\{X_i(x) | 1 \leq i \leq k\}$ where $k = k(x)$ depends on x (equivalently, each element of $\mathcal{D}$ is a value of a local section). An *integral manifold* F of D is a connected immersed submanifold of M such that for each $x \in F$,

$$T_xF = D_x.$$

A *C^∞-foliation* of M (in a generalized sense) is a C^∞-distribution $\mathcal{D}$ on M with the condition that for each point $x \in M$, there exists an integral manifold through x.

If $f : M \to N$ is a diffeomorphism of C^∞-manifolds M and N, and X a C^∞-vector field on N, then a C^∞-vector field f_*X on M is defined by

$$(f_*X)(y) = f_*(X(f^{-1}(y))) \qquad y \in N.$$

The integrability conditions are stated as follows in the formulation by P.Dazord [D].

THEOREM 1.1. *[Su], [D]. Let D be a C^∞-distrubution on a manifold M. The following properties are equivalent.*

(i) For each point $x \in M$, there exists an integral manifold through x.

(ii) For each point $x \in M$, there exists a unique maximal integral manifold through x.

(iii) D is invariant by the flow of each vector field of $C^\infty(D)$.

(iv) There is a Lie subalgebra $\mathcal{H}$ contained in the Lie algebra of C^∞-vector fields of M such that

a) at each point x, D_x is equal to the values at x of the fields in $\mathcal{H}$.

b) for each $X \in \mathcal{H}$ with a flow germ φ_t and for each $Y \in \mathcal{H}$, $(\varphi_t)_*Y$ is a germ of vector field of $\mathcal{H}$.

Let $\mathcal{D}$ be a C^∞-foliation of M. By THEOREM 1.1, (ii), for each point $x \in M$, there exists a unique maximal integral manifold through x, which we call a *leaf* of $\mathcal{D}$. Local structures of $\mathcal{D}$ are examined on the basis of THEOREM 1.1.

THEOREM 1.2. *[D]. For every point m of M with a C^∞-foliation $\mathcal{D}$, there is an open neighborhood U of x and a local coordinate map $\psi : U \xrightarrow{\cong} W \times V$ as follows: W and V are neighborhoods of origins in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively for*

$$p = \rho(m), \qquad p + q = \dim M,$$

and ψ carries the foliation induced on U to the product foliation,

$$\mathbb{R}^p \times \Delta_V,$$

where Δ_V is a foliation on V with a point leaf at the origin of $\mathbb{R}^q$ and $\mathbb{R}^p$ is the one leaf foliation.

We call U a *foliation coordinate neighborhood* and ψ a *local foliation coordinate system* (or *chart*). Also we call a leaf of $\mathcal{D}_U$ a *plaque*. Foliation coordinate neighborhoods of a point form a fundamental system of neighborhoods. $\mathcal{D}_U$ is the pullback of Δ_V by the submersion $\pi_U = \pi \cdot \psi : U \to V$.

In general, plaques other than the leaf of m are not (locally closed) submanifolds of M. This is a difference from the regular case. From THEOREM 1.2, it follows that each plaque of U is diffeomorphic to the product of the plaque of m and a leaf of Δ_V. In particular, $\rho = \dim \mathcal{D}$ is lower semi-continuous.

Let M' be an immersed submanifold of M transverse to the foliation $\mathcal{D}$ and U an open foliation coordinate neighborhood of $m \in M'$. The submersion $\pi_U : U \to V$ associated with a local foliation chart ψ induces a submersion π'_U of the neighborhood $M'_U = U \cap M'$ of m in V and the restriction $\mathcal{D}|M'_U$ is the pullback $\pi'^{*}_U \Delta_V$. This shows from the definition of foliation that the restriction $\mathcal{D}_{M'}$ is a foliation.

COROLLARY 1.3. *[D]. Let M_1 and M_2 be two submanifolds of M passing through m and being transverse to $\mathcal{D}$ at m. If we have*

$$\dim\ M_2 = \dim\ M_1 + r \qquad (r \geq 0),$$

then the foliation $\mathcal{D}_{M_2}$ of M_2 is locally isomorphic, on a neighborhood of m to the product foliation $\mathbb{R}^r \times \mathcal{D}_{M_1}$, of $\mathbb{R}^r \times M_1$.

In particular, if $r = 0$, then $\mathcal{D}_{M_1}$ and $\mathcal{D}_{M_2}$ are locally isomorphic. If M' is the image of V in the local foliation chart ψ, then $\mathcal{D}_{M'}$ is isomorphic to Δ_V. This yields that the germ at the origin O, defined by Δ_V depends, up to a diffeomorphism, on the point m and it is called the *germ of transverse structure of* $\mathcal{D}$ at m. If $\rho(m) = 0$, then Δ_V defines just the germ of $\mathcal{D}$ at m.

2. Holonomy groupoids

Let M be a manifold with a C^∞-foliation $\mathcal{D}$. Since every point of the leaf F of $m \in M$ is attained starting from m by a product of flows tangent to $\mathcal{D}$, germs of transverse structures at all points of F are isomorphic, and therefore there is a unique germ Δ_F of transverse structure. It is called the *germ of transverse structure of the leaf* F.

A leaf F is called *regular* if its germ of transverse structure is trivial, and it is called *singular* otherwise. A leaf is regular if and only if it has a neighborhood on which $\mathcal{D}$ induces a regular foliation, or equivalently, ρ is constant. A point is called *regular* if it belongs to a regular leaf. We say that a singular leaf F is *tractable* if F has a saturated neighborhood N in M with following properties:

(i) N is isomorphic to a fibre bundle over F, $\pi_F : N \to F$ having a fibre V with a foliation Δ_V which is a representative of Δ_F.

(ii) The structural group of the bundle is the group of isomorphisms of Δ_V and the foliation of N determined by a local product of one leaf foliation of F and Δ_V is the restriction $\mathcal{D}_N$ of $\mathcal{D}$ to N.

In the following, we consider a foliation $\mathcal{D}$ whose singular leaves are all tractable.

A tractable singular leaf F of a foliation $\mathcal{D}$ has a saturated tubular neighborhood which is isomorphic to a bundle with a fibre which is a foliated disk V. We take the associated V/Δ_V-bundle. Then any continuous curve of F obviously determines an isomorphism from the leaf space of fibre of the source point to that of the target point. Hence one can define the *holonomy map* with respect to the germ of V/Δ_V, associated with a curve on F by that isomorphism. By the elementary theory of fibre bundles this map is determined up to homotopy of the curves fixing end points. Under the assumption that each singular leaf of C^∞-foliation $\mathcal{D}$ is tractable, one can construct a holonomy groupoid $G(\mathcal{D})$ by quite a similar way to the regular case.

For a leaf L of $\mathcal{D}$, let

$$\lambda : [0,1] \to L$$

be a continuous curve with end points $\lambda(0) = x$ and $\lambda(1) = y$. Let V_m denote a sufficiently small manifold of dimension dim $M -$ dim L transverse to L at $m \in L$ and $H^\lambda_{x,y}$: $V_x/\Delta_{V_x} \to V_y/\Delta_{V_y}$ the holonomy map germ associated with λ, which depends only on the homotopy class $\bar{\lambda}$ relative to $\{0,1\}$.

Let $\mu : [0,1] \to L$ be another curve with $\mu(0) = x$ and $\mu(1) = y$, and μ^{-1} its inverse curve. We define a relation $\lambda \sim \mu$ by the equation,

$$H^{\lambda \cdot \mu^{-1}}_{x,x} = id.$$

This is an equivalence relation; the equivalence class of λ is denoted by $[\lambda]$.

Let $G(\mathcal{D})$ be the set of triples,

$$g = (x, y, [\lambda]),$$

where $x, y \in L$, L is a leaf of $\mathcal{D}$, and $\lambda : [0,1] \to L$ is a continuous curve with $\lambda(0) = x$ and $\lambda(1) = y$. The maps $s, r : G(\mathcal{D}) \to M$ are defined by

$$s((x, y, [\lambda])) = x, \qquad\qquad r((x, y, [\lambda])) = y$$

and are called *source* and *target* map respectively. $G(\mathcal{D})$ is a groupoid over M by the usual composition and inverse operations obtained from those of curves:

$$(x, y, [\lambda_1]) \cdot (y, z, [\lambda_2]) = (x, z, [\lambda_1 \cdot \lambda_2]),$$
$$(x, y, [\lambda])^{-1} = (y, x, [\lambda^{-1}]),$$

where λ^{-1} is the inverse curve of λ.

We will introduce a topology in $G(\mathcal{D})$ by defining fundamental systems of neighborhoods of points. For a point belonging to a regular leaf, its neighborhoods are the same as in the case of a regular foliation: For $g = (s, y, [\lambda])$, there exist a sequence of foliation coordinate neighborhoods $\{U_i | 0 \le i \le k\}$ and a partition of $[0,1]$, $0 = t_0 < \cdots < t_{k+1} = 1$ such that if $U_i \cap U_j \ne \phi$ then $U_i \cup U_j$ is contained in a foliation coordinate neighborhood and $\lambda([t_i, t_{i+1}]) \subset U_i$ for all $0 \le i < k+1$. We call the sequence $\{U_i | 0 \le i \le k\}$ a *chain subordinated* to λ. A neighborhood of g is the set $U_{g,\lambda}$ of $(x', y', [\nu])$ such that $x' \in U_0$, $y' \in U_k$ and $\{U_i | 0 \le i \le k\}$ is a chain subordinated to ν.

Let $a = (u, v, [\lambda])$ be a point of $G(\mathcal{D})$ such that $\lambda([0,1])$ is contained in a singular leaf F, which is, of course, tractable by our assumption. Let $\{U_i^! | 0 \leq i \leq k\}$ be a sequence of open sets in F, which is a chain subordinated to λ for one leaf foliation. Let Δ_V be a representative of Δ_F in a transverse submanifold V of dimension, dim M − dim F in M. We note that $\pi_F^{-1}(U_i^!) \cong U_i^! \times V$ and the sequence $\{\pi_F^{-1}(U_i^!) | 0 \leq i \leq k\}$ is *a chain subordinated to* λ for $\mathcal{D}$ in a generalized sense. We define a neighborhood of a in $G(\mathcal{D})$ by the set $U_{a,\lambda}$ of $(u', v', [\nu])$ such that $u' \in \pi_F^{-1}(U_0^!)$, $v' \in \pi_F^{-1}(U_k^!)$ and $\{\pi_F^{-1}(U_i^!) | 0 \leq i \leq k\}$ is a chain subordinated to ν.

THEOREM 2.1. *If $\mathcal{D}$ is a C^∞-foliation of M and each singular leaf of $\mathcal{D}$ is tractable, then $G(\mathcal{D})$ is a topological groupoid.*

3. EXAMPLES FROM POISSON STRUCTURES

Important examples of generalized foliations are theose of symplectic leaves in Poisson manifolds. Let M be a C^∞-manifold and $C^\infty(M)$ an algebra of real valued C^∞-function on M. A Poisson structure on M is defined as a Lie algebra structure $\{\ ,\ \}$ on $C^\infty(M)$ satisfying the Leibnitz identity,

$$\{fg, h\} = f\{g, h\} + \{f, h\}g.$$

The manifold M equipped with such a structure is called a *Poisson manifold.*

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. For $f, g \in C^\infty(\mathfrak{g}^*, \mathbb{R})$, we set

$$\{f, g\}(x) = < x, [df(x), dg(x)] > .$$

This gives a Poisson structure on $\mathfrak{g}^*$, which was defined by S. Lie and F. A. Berezin. Each leaf of the C^∞-foliation of $\mathfrak{g}*$ associated with this Poisson structure is a coadjoint orbit of G. (See, e.g., [K2], [Ko] and [So].)

We will examine our holonomy groupoids of foliations of $\mathfrak{g}^*$ by coadjoint orbits of G for some Lie groups mentioned in A.Weinstein [We]. It is noted that the holonomy groupoid of the foliation outside singular leaves is a (non-Hausdorff) manifold.

Let $\{X_i | 1 \leq i \leq n\}$ be a basis of $\mathfrak{g}$ with $n = \dim\ \mathfrak{g}$ and $x_1, \ldots, x_n$ the linear functions corresponding to these basis elements.

EXAMPLE 3.1: $G = SO(3)$. One can take a basis $\{X_1, X_2, X_3\}$ of the Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$ such that Poisson brackets of x_i are given by

$$\{x_1, x_2\} = x_3, \quad \{x_2, x_3\} = x_1, \quad \{x_3, x_1\} = x_2.$$

The manifold M is $\mathfrak{so}(3)^* \cong \mathbf{R}^3$. Leaves of $\mathcal{D}$ are coadjoint orbits of $SO(3)$ in $\mathfrak{so}(3)^*$ which are concentric spheres:

$${x_1}^2 + {x_2}^2 + {x_3}^2 = c > 0, \quad c \in \mathbf{R}.$$

The origin $O = (0,0,0) \in \mathbf{R}^3$ is the only singular leaf which is obviously tractable and all holonomy maps are trivial.

The holonomy groupoid $G(\mathcal{D})$ of the foliation $\mathcal{D}$ is described as follows: The holonomy groupoid of the regular part of $\mathcal{D}$ is

$$S^2 \times S^2 \times (\mathbf{R}_+ \backslash \{0\})$$

and hence $G(\mathcal{D})$ is the cone over $S^2 \times S^2$,

$$C(S^2 \times S^2) \cong (S^2 \times S^2 \times \mathbf{R}_+)/(S^2 \times S^2 \times \{0\}).$$

EXAMPLE 3.2: $G = SL(2, \mathbf{R})$. One can take a basis $\{X_1, X_2, X_3\}$ of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R})$ such that Poisson brackets of x_i are given by

$$\{x_1, x_2\} = -x_3, \quad \{x_2, x_3\} = x_1, \quad \{x_3, x_1\} = x_2.$$

The manifold M is $\mathfrak{sl}(2,\mathbb{R})^* \cong \mathbb{R}^3$. Leaves of $\mathcal{D}$ are coadjoint orbits of $SL(2,\mathbb{R})$ in $\mathfrak{sl}(2,\mathbb{R})^*$ which are the origin, one sheet hyperboloids, two sheet hyperboloids and circular cones:

$$\{(0,0,0)\},$$
$$x_1{}^2 + x_2{}^2 - x_3{}^2 = c \neq 0, \quad c \in \mathbb{R},$$
$$x_1{}^2 + x_2{}^2 - x_3{}^2 = 0, \quad x_3 \neq 0.$$

The origin $O = (0,0,0)$ is the only singular leaf, which is obviously tractable, and all holonomy maps are trivial.

The holonomy groupoid $G(\mathcal{D})$ of the foliation $\mathcal{D}$ is described as follows: The holonomy groupoid of the regular part of $\mathcal{D}$ is the disjoint union with an appropriate topology,

$$(S^1 \times \mathbb{R} \times S^1 \times \mathbb{R} \times \mathbb{R}) \cup (B^2 \times B^2 \times \mathbb{R}^+) \cup (B^2 \times B^2 \times \mathbb{R}^-)$$
$$\cup (S^1 \times \mathbb{R}^+ \times S^1 \times \mathbb{R}^+) \cup (S^1 \times \mathbb{R}^- \times S^1 \times \mathbb{R}^-),$$

where B^2 is the open 2-disk and $\mathbb{R}^\pm = \mathbb{R}_\pm \backslash \{0\}$. The holonomy groupoid $G(\mathcal{D})$ is the disjoint union with an appropriate topology,

$$(S^1 \times \mathbb{R} \times S^1 \times \mathbb{R} \times \mathbb{R}) \cup (B^2 \times B^2 \times \mathbb{R}^+) \cup (B^2 \times B^2 \times \mathbb{R}^-) \cup (S^1 \times \mathbb{R}^+ \times S^1 \times \mathbb{R}^+)$$
$$\cup (S^1 \times \mathbb{R}^- \times S^1 \times \mathbb{R}^-) \cup *$$

where $*$ is the only element of $G(\mathcal{D})$ obtained from the leaf O.

4. Various examples

First of all, we mention one more example of a generalized foliation with trivial holonomy maps, which is not locally simple.

EXAMPLE 4.1: Let $S_{\lambda,\mu}$ be a circle in $\mathbf{R}^2$:

$$\lambda({x_1}^2 + {x_2}^2 - 2x_2) + \mu x_2 = 0$$
$$\lambda, \mu \in \mathbf{R}.$$

The set $\{S_{\lambda,\mu} | \lambda, \mu \in \mathbf{R}\}$ defines a generalized foliation $\mathcal{D}$ of $\mathbf{R}^2$ with $D_O = \{0\}$. $\mathcal{D}$ has the only singular leaf $\{O\}$.

A foliation of a manifold M is *locally simple* by definition (see [E]), if each point x of M has an open neighborhood V such that for the fundamental system of open neighborhoods U of x in V, the maps of the leaf spaces $\tilde{U}$ to the leaf space $\tilde{V}$ of V, induced by inclusion maps, are homeomorphisms onto open sets of $\tilde{V}$. In our foliation $\mathcal{D}$, it is obvious that the origin $O \in \mathbf{R}^2 = M$ does not satisfy the condition of local simplicity. In fact, for any fundamental system of neighborhoods of O contains open sets U, V such that $U \subsetneqq V$ and the map $\tilde{U} \to \tilde{V}$ is not injective.

The holonomy groupoid $G(\mathcal{D}|_{\mathbf{R}^3\setminus\{O\}})$ of the regular part of $\mathcal{D}$ is diffeomorphic to the manifold

$$((S^1\setminus\{q\})\times(S^1\setminus\{q\})\times(S^1\setminus\{q,-q\}))\cup((S^1\setminus\{q,-q\})\times(S^1\setminus\{q,-q\}\times\{-q\})) \subset S^1\times S^1\times S^1,$$

where S^1 is the unit circle and $q = (0,1) \in S^1$. In the quotient space $Q = (S^2 \times S^1 \times S^1)/((S^1 \times S^1 \times \{q\}) \cup (\{q\} \times \{q\} \times S^1))$, we denote the point $[\{q\} \times \{q\} \times \{q\}]$ by $*$. Then $G(\mathcal{D})$ is the disjoint union

$$G(\mathcal{D}|_{\mathbf{R}^2\setminus\{O\}}) \cup *$$

and $*$ is the element represented by the point leaf $\{O\}$.

Examples in Section 3 obtained from foliations of coadjoint orbits of Lie groups in the dual of its Lie algebra, have all trivial holonomy maps. However, some generalized foliations with nontrivial holonomy maps are constructed as follows:

EXAMPLE 4.2: Let $\mathcal{D}$ be the same foliations of $\mathbb{R}^3$ as in EXAMPLES 3.1 and 3.2. We define a C^∞-diffeomorphism $f : \mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}^3$ by

$$f(x_1, x_2, x_3) = e(x_1, x_2, x_3),$$

where $e > 1$. We identify points of $\mathbb{R}^3 \times \{0\}$ in $\mathbb{R}^3 \times [0,1]$ to points of $\mathbb{R}^3 \times \{1\}$ by the diffeomerphism

$$x_1, x_2, x_3, 0) \mapsto (f(x_1, x_2, x_3), 1).$$

Since f preserves the foliation $\mathcal{D}$, one obtains a foliation $\mathcal{D}_S$ on $S = \mathbb{R}^3 \times S^1$ from $\mathcal{D}$ by taking a local product with the one leaf foliation of $\mathbb{R}$. The identification image F_S of $\{O\} \times [0,1]$ is the only singular leaf of $\mathcal{D}_S$. The asociated $\mathbb{R}^3/\mathcal{D}$-bundle over F_S is flat and its holonomy group is $\mathbb{Z}$ which is again nontrivial.

We define a homomorphism $h_S : \pi_1(S^1 \times S^1) \to \mathbb{Z}$ by

$$h_S(\alpha) = -h_S(\beta) = 1 \in \mathbb{Z}.$$

Let K be the set of continuous curves $\gamma : [0,1] \to S^1$. For $\gamma_1, \gamma_2 \in K$, we define a relation $\gamma_1 \sim \gamma_2$ if we have

$$\gamma_1(0) = \gamma_2(0), \quad \gamma_1(1) = \gamma_2(1)$$

and

$$h_S([\gamma_1 \cdot \gamma_2^{-1}]) = 0,$$

where γ_2^{-1} is the inverse curve of γ_2 , $\gamma_1 \cdot \gamma_2^{-1}$ is the composition of curves and $[\gamma_1 \cdot \gamma_2^{-1}]$ is the homotopy class of $\gamma_1 \cdot \gamma_2^{-1}$. This relation is an equivalence relation and we denote the quotient space of K by $\overline{K}_S = K/\sim$ which is diffeomorphic to an open cylinder.

Since $\mathbf{Z}$ is a group of diffeomorphisms of $\mathbf{R}^3$ preserving the foliation $\mathcal{D}_S$, it acts on the manifold $G(\mathcal{D}|_{\mathbf{R}^3\setminus\{O\}})$ and hence $\pi_1(S^1 \times S^1) \cong \mathbf{Z}^2$ acts on $G(\mathcal{D}|_{\mathbf{R}^3\setminus\{O\}})$ through the homomorphism h_S, the holonomy groupoid of the regular part $\mathcal{D}|_{S\setminus F_S}$ is the bundle associated with the standard covering map $\mathbf{R}^2 \to S^1 \times S^1$:

$$G(\mathcal{D}_S|_{S\setminus F_S}) = G(\mathcal{D}|_{\mathbf{R}^3\setminus\{O\}}) \times_{\mathbf{Z}^2} \mathbf{R}^2$$

The holonomy groupoid $G(\mathcal{D}_S)$ is of the form of disjoint union

$$(G(\mathcal{D}_S|_{S\setminus F_S}) \cup \overline{K_S}$$

with an appropriate topology.

A similar construction is made with the foliation $\mathcal{D}$ of EXAMPLE 4.1 and a holonomy groupoid of a locally nonsimple foliation is obtained.

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On Hypersurface Simple K3 Singularities

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Introduction

In the theory of 2-dimensional singularities, simple elliptic singularities and cusp singularities are considered as the next most reasonable class of singularities after rational singularities. Cusp singularities appear on the Satake compactification of Hilbert modular surfaces, and have a loop of rational curves as the exceptional set of the minimal resolution. Simple elliptic singularities are investigated by K. Saito[11] in detail. By definition, they have a non-singular elliptic curve as the exceptional set of the minimal resolution. Here we are interested especially in a hypersurface simple elliptic singularity (X,x). In this case, the defining equation of (X,x) is given by one of the followings in some coordinates z_1, z_2, z_3 around x.

$$\tilde{E}_6 : \quad z_1^3 + z_2^3 + z_3^3 + \lambda z_1 z_2 z_3 = 0 \qquad (E^2 = -3),$$

$$\tilde{E}_7 : \quad z_1^2 + z_2^4 + z_3^4 + \lambda z_1 z_2 z_3 = 0 \qquad (E^2 = -2),$$

$$\tilde{E}_8 : \quad z_1^2 + z_2^3 + z_3^6 + \lambda z_1 z_2 z_3 = 0 \qquad (E^2 = -1).$$

The parameter λ corresponds to the moduli of the elliptic curve E which appears as the exceptional set.

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[1]This is a preliminary version.

The purpose of this paper is to study similar properties for a simple K3 singularity which we regard as a natural generalization of a simple elliptic singularity in 3-dimensional case.

The notion of a simple K3 singularity is defined by K. Watanabe[4] as a 3-dimensional Gorenstein purely elliptic singularity of (0,2)-type. (Note that a simple elliptic singularity is a 2-dimensional purely elliptic singularity of (0,1)-type.) S. Ishii[4] pointed out that simple K3 singularities are characterized as quasi-Gorenstein singularities whose exceptional set of any minimal resolution is a normal K3 surface. Let $f \in \mathbb{C}[z_0, z_1, z_2, z_3]$ be a polynomial which is non-degenerate with respect to its Newton boundary $\Gamma(f)$ in the sence of [14], and whose zero locus $X = \{f=0\}$ in $\mathbb{C}^4$ has an isolated singular point at the origin $0 \in \mathbb{C}^4$. Then the condition for $(X,0)$ to be a simple K3 singularity is given by a property of the Newton boundary $\Gamma(f)$ of f (cf. Proposition 1.6.). M. Tomari[12] showed that the minimal resolution $\pi:(\tilde{X},E) \longrightarrow (X,0)$ of a simple K3 singularity is also obtained from $\Gamma(f)$. In this paper, we classify non-degenerate hypersurface simple K3 singularities and study the singularities on the K3 surface E through the minimal resolution π.

Now we will explain the content of each section.

In §2, we classify non-degenerate hypersurface simple K3 singularities into 95 classes in terms of the "weight" of f.

In §3, we construct the minimal resolution π using the method of torus embeddings, and study the singularities on the weighted projective space $\mathbb{P}(p_1, p_2, p_3, p_4)$ for the next section.

In §4, we prove that the singularities on the normal K3 surface

E are determined by the "weight" of f, and show the relation between the rank of singularities on E and the number of "parameters" in f.

Notation

We denote by $\mathbb{R}_0$ (resp. $\mathbb{R}_+$) the set of all positive (resp. non-negative) real numbers. Similarly we define $\mathbb{Q}_0$, $\mathbb{Q}_+$, $\mathbb{Z}_0$, $\mathbb{Z}_+$ etc.

§1. Preliminaries

In this section, we recall some definitions and results from [2],[4],[15] and [16].

First we define the plurigenera δ_m ($m\in\mathbb{N}$) for normal isolated singularities and define purely elliptic singularities. Let (X,x) be a normal isolated singularity in the n-dimensional analytic space X , and $\pi:(\tilde{X},E)\longrightarrow(X,x)$ a good resolution. In the following, if necessary, we assume that X is a sufficiently small Stein neighbourhood of x .

Definition 1.1(K. Watanabe[15]). Let (X,x) be a normal isolated singularity. For any positive integer m ,

$$\delta_m(X,x) := \dim_{\mathbb{C}}\Gamma(X-\{x\},\mathcal{O}(mK)/L^{2/m}(X-\{x\})$$

where K is the canonical line bundle on $X-\{x\}$, and $L^{2/m}(X-\{x\})$ is the set of all $L^{2/m}$-integrable holomorphic m-ple n-forms on $X-\{x\}$.

Then δ_m is finite and does not depend on the choice of Stein neighbourhood X .

Definition 1.2(K. Watanabe[15]). A singularity (X,x) is called purely elliptic if $\delta_m=1$ for any $m\in\mathbb{N}$.

When X is a 2-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein, i.e., there exists a non-vanishing holomorphic 2-form on $X-\{x\}$ [3]. But in higher dimensional case, purely elliptic singularities are not always quasi-Gorenstein. In the following, we assume that (X,x) is quasi-Gorenstein. Let $E = \bigcup E_i$ be the decomposition of the exceptional set E into irreducible components, and write $K_{\tilde{X}} = f^*K_X + \sum_{i\in I} m_i E_i - \sum_{j\in J} m_j E_j$, where $m_i \geq 0$, $m_j > 0$. S. Ishii defined in [2] the essential part of the exceptional set E as $E_J = \sum_{j\in J} m_j E_j$, and showed that if (X,x) is purely elliptic, then $m_j=1$ for all $j\in J$.

Definition 1.3(S. Ishii [2]). A quasi-Gorenstein purely elliptic singularity (X,x) is of $(0,i)$-type if $H^{n-1}(E_J,\mathcal{O}_E)$ consists of $(0,i)$-Hodge component $H^{0,i}(E_J)$, where

$$\mathbb{C} \simeq H^{n-1}(E_J,\mathcal{O}_E) = Gr_F^0 H^{n-1}(E_J) = \bigoplus_{i=0}^{n-1} H^{0,i}(E_J).$$

Definition-Proposition 1.4(K. Watanabe-S. Ishii [4]).
A 3-dimensional singularity (X,x) is a simple K3 singularity if the following two equivalent conditions are satisfied:

(1) (X,x) is Gorenstein purely elliptic of $(0,2)$-type.

(2) (X,x) is quasi-Gorenstein and exceptional divisor E is a normal K3 surface for any minimal resolution $\pi:(\tilde{X},E)\longrightarrow(X,x)$.

Remark 1.5. A minimal resolution $\pi:(\tilde{X},E)\longrightarrow(X,x)$ is a proper morphism with $\tilde{X}-E \simeq X-\{x\}$ where $\tilde{X}$ has only terminal singularities and $K_{\tilde{X}}$ is numerically effective with respect to π .

Next we consider the case that (X,x) is a hypersurface singularity which is defined by a non-degenerate polynomial $f = \Sigma a_\nu z^\nu \in \mathbb{C}[z_0,z_1,\cdots,z_n]$, and $x=0\in\mathbb{C}^{n+1}$. Recall that the Newton boundary $\Gamma(f)$ of f is the union of the compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of $\bigcup_{a_\nu\neq 0}(\nu+\mathbb{R}_0^{n+1})$ in $\mathbb{R}^{n+1}$. For any face Δ of $\Gamma_+(f)$, set $f_\Delta := \sum_{\nu\in\Delta} a_\nu z^\nu$. We say f is non-degenerate if,

$$\frac{\partial f_\Delta}{\partial z_0} = \frac{\partial f_\Delta}{\partial z_1} = \cdots = \frac{\partial f_\Delta}{\partial z_n} = 0$$

has no solution in $(\mathbb{C}^*)^{n+1}$ for any face Δ . When f is non-degenerate, the condition for (X,x) to be a purely elliptic singularity is given as follows.

Theorem 1.6(K. Watanabe [16]). Let f be a non-degenerate polynomial and $X=\{f=0\}$ has an isolated singularity at $x=0\in\mathbb{C}^{n+1}$.

(1) (X,x) is purely elliptic if and only if $(1,1,\cdots,1)\in\Gamma(f)$.

(2) Let $n=3$ and Δ_0 be the face of $\Gamma(f)$ which contains $(1,1,1,1)$ in the relative interior of Δ_0 , then (X,x) is simple K3 singularity if and only if $\dim_{\mathbb{R}}\Delta_0=3$.

So if f is non-degenerate and defines a simple K3 singularity, then f_{Δ_0} is quasi-homogeneous polynomial whose weight α is uniquely determined. In this case, we call α the weight of f and write $\alpha(f)$, i.e., $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in\mathbb{Q}_+^4$ and $\deg_\alpha(\nu):=\sum_{i=1}^{4}\alpha_i\nu_i=1$ for any $\nu\in\Delta_0$. In particular, $(1,1,1,1)$ is always contained in Δ_0 and $\sum_{i=1}^{4}\alpha_i=1$.

§2. Weights of hypersurface simple K3 singularities

In this section, we calculate weights of hypersurface simple K3 singularities defined by non-degenerate polynomials.

Let $W' := \{\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Q}_+^4 \mid \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1\}$ and for an element α of W', set $T(\alpha) := \{\nu \in \mathbb{Z}_0^4 \mid \alpha \cdot \nu = 1\}$ and $\langle T(\alpha) \rangle := \{\sum_{\nu \in T(\alpha)} t_\nu \cdot \nu \in \mathbb{R}^4 \mid t_\nu \in \mathbb{R}_0\}$. Then the set $\langle T(\alpha) \rangle$ is a closed cone in $\mathbb{R}^4$ spanned by $T(\alpha)$.

Let $W_4 := \{\alpha \in W' \mid (1,1,1,1) \in \mathrm{Int}\langle T(\alpha) \rangle,\ \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4\}$.

By Theorem 1.6., W_4 is the set of weights of simple K3 singularities.

Proposition 2.1. $\# W_4 = 95$.

Before proving this proposition, we give the complete list of weights $\alpha \in W_4$ and examples of $f = \sum_{\nu \in T(\alpha)} a_\nu z^\nu$ which is quasi-homogeneous and $\{f=0\} \subset \mathbb{C}^4$ has a simple K3 singularity at the origin $0 \in \mathbb{C}^4$. The polynomials f in Table 2.2. are chosen to satisfy the condition that $a_\nu \neq 0$ if and only if ν is a extremal point of the convex hull of $T(\alpha)$ in $\mathbb{R}^4$, in particular $\Gamma(f) = \Delta_0$ is the convex hull of $T(\alpha)$.

Table 2.2.

No.	weight α	f	#T(α)
1	$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$x^4+y^4+z^4+w^4$	35
2	$(\frac{1}{3},\frac{1}{4},\frac{1}{4},\frac{1}{6})$	$x^3+y^4+z^4+w^6$	15
3	$(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6})$	$x^3+y^3+z^6+w^6$	30
4	$(\frac{1}{3},\frac{1}{3},\frac{1}{4},\frac{1}{12})$	$x^3+y^3+z^4+w^{12}$	21
5	$(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6})$	$x^2+y^6+z^6+w^6$	39
6	$(\frac{1}{2},\frac{1}{5},\frac{1}{5},\frac{1}{10})$	$x^2+y^5+z^5+w^{10}$	28
7	$(\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{8})$	$x^2+y^4+z^8+w^8$	35
8	$(\frac{1}{2},\frac{1}{4},\frac{1}{6},\frac{1}{12})$	$x^2+y^4+z^6+w^{12}$	27
9	$(\frac{1}{2},\frac{1}{4},\frac{1}{5},\frac{1}{20})$	$x^2+y^4+z^5+w^{20}$	23
10	$(\frac{1}{2},\frac{1}{3},\frac{1}{12},\frac{1}{12})$	$x^2+y^3+z^{12}+w^{12}$	39
11	$(\frac{1}{2},\frac{1}{3},\frac{1}{10},\frac{1}{15})$	$x^2+y^3+z^{10}+w^{15}$	18
12	$(\frac{1}{2},\frac{1}{3},\frac{1}{9},\frac{1}{18})$	$x^2+y^3+z^9+w^{18}$	30
13	$(\frac{1}{2},\frac{1}{3},\frac{1}{8},\frac{1}{24})$	$x^2+y^3+z^8+w^{24}$	27
14	$(\frac{1}{2},\frac{1}{3},\frac{1}{7},\frac{1}{42})$	$x^2+y^3+z^7+w^{42}$	24
15	$(\frac{1}{3},\frac{4}{15},\frac{1}{5},\frac{1}{5})$	$x^3+y^3z+y^3w+z^5-w^5$	12
16	$(\frac{1}{3},\frac{7}{24},\frac{1}{4},\frac{1}{8})$	$x^3+y^3w+z^4+w^{18}$	9
17	$(\frac{1}{3},\frac{1}{3},\frac{1}{5},\frac{2}{15})$	$x^3+y^3+z^5+xw^5+yw^5+zw^6$	14
18	$(\frac{1}{3},\frac{1}{3},\frac{2}{9},\frac{1}{9})$	$x^3+y^3+xz^3+yz^3+z^4w+w^9$	23
19	$(\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{1}{8})$	$x^2y+x^2z+x^2w^2+y^4+z^4+w^8$	24
20	$(\frac{3}{8},\frac{1}{3},\frac{1}{4},\frac{1}{24})$	$x^2z+x^2w^6+y^3+z^4+w^{24}$	18
21	$(\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5})$	$x^2y+x^2z+x^2w+y^5+z^5+w^5$	34
22	$(\frac{2}{5},\frac{1}{3},\frac{1}{5},\frac{1}{15})$	$x^2z+x^2w^3+y^3+z^5-w^{15}$	21
23	$(\frac{5}{12},\frac{1}{4},\frac{1}{6},\frac{1}{6})$	$x^2z+x^2w+y^4+z^6+w^6$	17
24	$(\frac{5}{12},\frac{1}{3},\frac{1}{6},\frac{1}{12})$	$x^2z+x^2w^2+y^3+z^6+w^{12}$	24
25	$(\frac{4}{9},\frac{1}{3},\frac{1}{9},\frac{1}{9})$	$x^2z+x^2w+y^3+z^9-w^9$	33

26	$(\frac{9}{20},\frac{1}{4},\frac{1}{5},\frac{1}{10})$	$x^2w+y^4+z^5+w^{10}$	13
27	$(\frac{11}{24},\frac{1}{3},\frac{1}{8},\frac{1}{12})$	$x^2w+y^3+z^8+w^{12}$	15
28	$(\frac{10}{21},\frac{1}{3},\frac{1}{7},\frac{1}{21})$	$x^2w+y^3+z^7+w^{21}$	24
29	$(\frac{1}{2},\frac{1}{5},\frac{1}{6},\frac{2}{15})$	$x^2+y^5+z^6+yw^6+z^2w^5$	10
30	$(\frac{1}{2},\frac{1}{5},\frac{7}{40},\frac{1}{8})$	$x^2+y^5+z^5w+w^8$	8
31	$(\frac{1}{2},\frac{5}{24},\frac{1}{6},\frac{1}{8})$	$x^2+y^4z+y^3w^3+z^6+w^8$	12
32	$(\frac{1}{2},\frac{3}{14},\frac{1}{7},\frac{1}{7})$	$x^2+y^4z+y^4w+z^7-w^7$	19
33	$(\frac{1}{2},\frac{2}{9},\frac{1}{6},\frac{1}{9})$	$x^2+y^3z^2+y^4w+z^6+w^9$	16
34	$(\frac{1}{2},\frac{7}{30},\frac{1}{5},\frac{1}{15})$	$x^2+y^4w+z^5+w^{15}$	13
35	$(\frac{1}{2},\frac{1}{4},\frac{1}{7},\frac{3}{28})$	$x^2+y^4+z^7+yw^7+zw^8$	12
36	$(\frac{1}{2},\frac{1}{4},\frac{3}{20},\frac{1}{10})$	$x^2+y^4+yz^5+z^6w+w^{10}$	16
37	$(\frac{1}{2},\frac{1}{4},\frac{3}{16},\frac{1}{16})$	$x^2+y^4+yz^4+z^5w+w^{16}$	24
38	$(\frac{1}{2},\frac{4}{15},\frac{1}{5},\frac{1}{30})$	$x^2+y^3z+y^3w^6+z^5-w^{30}$	21
39	$(\frac{1}{2},\frac{5}{18},\frac{1}{6},\frac{1}{18})$	$x^2+y^3z+y^3w^3+z^6+w^{18}$	24
40	$(\frac{1}{2},\frac{2}{7},\frac{1}{7},\frac{1}{14})$	$x^2+y^3z+y^3w^2+z^7-w^{14}$	27
41	$(\frac{1}{2},\frac{7}{24},\frac{1}{8},\frac{1}{12})$	$x^2+y^3z+y^2w^5+z^8+w^{12}$	16
42	$(\frac{1}{2},\frac{3}{10},\frac{1}{10},\frac{1}{10})$	$x^2+y^3z+y^3w+z^{10}+w^{10}$	36
43	$(\frac{1}{2},\frac{11}{36},\frac{1}{9},\frac{1}{12})$	$x^2+y^3w+z^9+w^{12}$	12
44	$(\frac{1}{2},\frac{5}{16},\frac{1}{8},\frac{1}{16})$	$x^2+y^2z^3+y^3w+z^8+w^{16}$	28
45	$(\frac{1}{2},\frac{9}{28},\frac{1}{7},\frac{1}{28})$	$x^2+y^3w+z^7+w^{28}$	24
46	$(\frac{1}{2},\frac{1}{3},\frac{1}{11},\frac{5}{66})$	$x^2+y^3+z^{11}+zw^{12}$	9
47	$(\frac{1}{2},\frac{1}{3},\frac{2}{21},\frac{1}{14})$	$x^2+y^3+yz^7+z^9w^2+w^{14}$	13
48	$(\frac{1}{2},\frac{1}{3},\frac{5}{48},\frac{1}{16})$	$x^2+y^3+z^9w+w^{16}$	12
49	$(\frac{1}{2},\frac{1}{3},\frac{5}{42},\frac{1}{21})$	$x^2+y^3+z^8w+w^{21}$	15
50	$(\frac{1}{2},\frac{1}{3},\frac{2}{15},\frac{1}{30})$	$x^2+y^3+yz^5+z^7w^2+w^{30}$	25
51	$(\frac{1}{2},\frac{1}{3},\frac{5}{36},\frac{1}{36})$	$x^2+y^3+z^7w+w^{36}$	24
52	$(\frac{1}{3},\frac{1}{4},\frac{2}{9},\frac{7}{36})$	$x^3+y^4+xz^3+zw^4$	5

53	$(\frac{1}{3},\frac{5}{18},\frac{2}{9},\frac{1}{6})$	$x^3+y^3w+y^2z^2+xz^3+z^3w^2+w^6$	10
54	$(\frac{1}{3},\frac{2}{7},\frac{5}{21},\frac{1}{7})$	$x^3+y^3w+yz^3+z^3w^2-w^7$	9
55	$(\frac{7}{20},\frac{3}{10},\frac{1}{4},\frac{1}{10})$	$x^2y+x^2w^3+y^3w+z^4-w^{10}$	11
56	$(\frac{11}{30},\frac{4}{15},\frac{1}{5},\frac{1}{6})$	$x^2y+y^3z+z^5+w^6$	6
57	$(\frac{3}{8},\frac{1}{4},\frac{5}{24},\frac{1}{6})$	$x^2y+y^4+xz^3+z^4w+w^6$	8
58	$(\frac{3}{8},\frac{5}{16},\frac{1}{4},\frac{1}{16})$	$x^2z+x^2w^4+xy^2+y^3w+z^4+w^{16}$	19
59	$(\frac{8}{21},\frac{1}{3},\frac{5}{21},\frac{1}{21})$	$x^2z+x^2w^5+y^3+z^4w-w^{21}$	18
60	$(\frac{7}{18},\frac{1}{3},\frac{2}{9},\frac{1}{18})$	$x^2z+x^2w^4+y^3+yz^3+z^4w^2+w^{18}$	19
61	$(\frac{11}{28},\frac{1}{4},\frac{3}{14},\frac{1}{7})$	$x^2z+y^4+z^4w+w^7$	7
62	$(\frac{2}{5},\frac{1}{4},\frac{1}{5},\frac{3}{20})$	$x^2z+xw^4+y^4+yw^5+z^5+z^2w^4$	10
63	$(\frac{2}{5},\frac{3}{10},\frac{1}{5},\frac{1}{10})$	$x^2z+x^2w^2+xy^2+y^2z^2+y^3w+z^5+w^{10}$	23
64	$(\frac{5}{12},\frac{7}{24},\frac{1}{6},\frac{1}{8})$	$x^2z+xy^2+y^3w+z^6+w^8$	10
65	$(\frac{14}{33},\frac{1}{3},\frac{5}{33},\frac{1}{11})$	$x^2z+y^3+z^6w+w^{11}$	9
66	$(\frac{3}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7})$	$x^2z+x^2w+xy^2+y^3z+y^3w+z^7+w^7$	31
67	$(\frac{3}{7},\frac{1}{3},\frac{1}{7},\frac{2}{21})$	$x^2z+xw^6+y^3+yw^7+z^7+zw^9$	14
68	$(\frac{13}{30},\frac{1}{3},\frac{2}{15},\frac{1}{10})$	$x^2z+y^3+yz^5+z^6w^2+w^{10}$	10
69	$(\frac{7}{16},\frac{1}{4},\frac{3}{16},\frac{1}{8})$	$x^2w+xz^3+y^4+yz^4+z^4w^2+w^8$	14
70	$(\frac{4}{9},\frac{5}{18},\frac{1}{6},\frac{1}{9})$	$x^2w+xy^2+y^3z+y^2w^4+z^6+w^9$	14
71	$(\frac{7}{15},\frac{4}{15},\frac{1}{5},\frac{1}{15})$	$x^2w+xy^2+y^3z+y^3w^3+z^5+w^{15}$	22
72	$(\frac{7}{15},\frac{1}{3},\frac{2}{15},\frac{1}{15})$	$x^2w+xz^4+y^3+yz^5+z^7w+w^{15}$	26
73	$(\frac{1}{2},\frac{1}{5},\frac{4}{25},\frac{7}{50})$	$x^2+y^5+yz^5+zw^6$	6
74	$(\frac{1}{2},\frac{7}{32},\frac{5}{32},\frac{1}{8})$	$x^2+y^4w+yz^5+z^4w^3+w^8$	9
75	$(\frac{1}{2},\frac{5}{22},\frac{2}{11},\frac{1}{11})$	$x^2+y^4w+y^2z^3+z^5w+w^{11}$	14
76	$(\frac{1}{2},\frac{3}{13},\frac{5}{26},\frac{1}{13})$	$x^2+y^4w+yz^4+z^4w^3+w^{13}$	13
77	$(\frac{1}{2},\frac{7}{26},\frac{5}{26},\frac{1}{26})$	$x^2+y^3z+y^3w^5+z^5w+w^{26}$	21
78	$(\frac{1}{2},\frac{3}{11},\frac{2}{11},\frac{1}{22})$	$x^2+y^3z+y^3w^4+yz^4+z^5w^2+w^{22}$	22
79	$(\frac{1}{2},\frac{9}{32},\frac{5}{32},\frac{1}{16})$	$x^2+y^3z+y^2w^7+z^6w+w^{16}$	13

80	$(\frac{1}{2},\frac{13}{44},\frac{5}{44},\frac{1}{11})$	$x^2+y^3z+z^8w+w^{11}$	9
81	$(\frac{1}{2},\frac{4}{13},\frac{3}{26},\frac{1}{13})$	$x^2+y^3w+yz^6+z^8w+w^{13}$	16
82	$(\frac{1}{2},\frac{7}{22},\frac{3}{22},\frac{1}{22})$	$x^2+y^3w+yz^5+z^7w+w^{22}$	25
83	$(\frac{1}{2},\frac{1}{3},\frac{5}{54},\frac{2}{27})$	$x^2+y^3+yw^9+z^{10}w+z^2w^{11}$	10
84	$(\frac{1}{3},\frac{7}{27},\frac{2}{9},\frac{5}{27})$	$x^3+xz^3+y^3z+yw^{14}+z^2w^3$	6
85	$(\frac{5}{14},\frac{2}{7},\frac{3}{14},\frac{1}{7})$	$x^2y+x^2w^2+xz^3+y^3w+y^2z^2+z^4w+w^7$	13
86	$(\frac{9}{25},\frac{7}{25},\frac{1}{5},\frac{4}{25})$	$x^2y+xw^4+y^3w+z^5+zw^5$	7
87	$(\frac{5}{13},\frac{4}{13},\frac{3}{13},\frac{1}{13})$	$x^2z+x^2w^3+xy^2+y^3w+yz^3+z^4w+w^{13}$	20
88	$(\frac{11}{27},\frac{1}{3},\frac{5}{27},\frac{2}{27})$	$x^2z+xw^8+y^3+yw^9+z^5w+zw^{11}$	11
89	$(\frac{5}{11},\frac{3}{11},\frac{2}{11},\frac{1}{11})$	$x^2w+xy^2+xz^3+y^3z+y^3w^2+yz^4+z^5w+w^{11}$	24
90	$(\frac{1}{2},\frac{7}{34},\frac{3}{17},\frac{2}{17})$	$x^2+y^4z+y^2w^5+z^5w+zw^7$	8
91	$(\frac{1}{2},\frac{4}{19},\frac{3}{19},\frac{5}{38})$	$x^2+y^4z+yz^5+yw^6+z^3w^4$	7
92	$(\frac{1}{2},\frac{11}{38},\frac{5}{38},\frac{3}{38})$	$x^2+y^3z+yw^9+z^7w+zw^{11}$	10
93	$(\frac{1}{2},\frac{5}{17},\frac{2}{17},\frac{3}{34})$	$x^2+y^3z+yz^6+yw^8+z^7w^2+zw^{10}$	11
94	$(\frac{7}{19},\frac{5}{19},\frac{4}{19},\frac{3}{19})$	$x^2y+xz^3+xw^4+y^3z+y^2w^3+z^4w+zw^5$	9
95	$(\frac{7}{17},\frac{5}{17},\frac{3}{17},\frac{2}{17})$	$x^2z+xy^2+xw^5+y^3w+yz^4+yw^6+z^5w+zw^7$	13

We express a weight α in W_4 as $\alpha=(\frac{p_1}{p},\frac{p_2}{p},\frac{p_3}{p},\frac{p_4}{p})$ where p,p_i are positive integers with $g.c.d.(p_1,p_2,p_3,p_4) = 1$.

Lemma 2.3.

(1) For any $i=1,2,3,4$, one of the followings is satisfied:

(a) $p_i | p$,

(b) $p_i | (p-p_j)$ for some $j \neq i$.

(2) $g.c.d.(p_i,p_j,p_k) = 1$ for all distinct i,j,k.

(3) Let $a_{ij}:=\text{g.c.d.}(p_i,p_j)$ $(i\neq j)$, then $a_{ij}|p$.

(4) If $p_i|p$ and $p_i|(p-p_j)$, then $a_{ij}=p_i$, $a_{ik}=a_{i\ell}=1$. Where we set $\{i,j,k,\ell\}=\{1,2,3,4\}$.

Proof.

(1) See Table 2.2.

(2) Since $\text{g.c.d.}(p_1,p_2,p_3,p_4)=1$, if $\text{g.c.d.}(a_i,a_j,a_k)=d>1$, then $\text{g.c.d.}(a_\ell,d)=1$, $\text{g.c.d.}(p,d)=1$. Thus for every $\nu\in T(\alpha)$, $\nu_\ell\geq 1$ and this contradicts to the condition $(1,1,1,1)\in \text{Int}\langle T(\alpha)\rangle$.

(3) If $p_i|p$ or $p_i|(p-p_j)$, then the assertion is clear. Thus by (1), we may consider the case $p_i|(p-p_k)$. Then $p_i|(p_j+p_\ell)$ and hence, we have $a_{ij}=1$ by (2).

(4) From the condition $p_i|(p-p_j)$, we have $p_i|(p_k+p_\ell)$, and hence $a_{ik}=a_{i\ell}=1$ by (2). The other assertion $a_{ij}=p_i$ follows from the fact $p_i|p_j$.

Q.E.D.

Proof of Proposition 2.1.

Let $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ be a weight in W_4, and let H_0 be the hyperplane in $\mathbb{R}^4$ which contains $T(\alpha)$. We denote by δ the point $(1,1,1,1)$ in $T(\alpha)$.

First, we explain the outline of the proof. By definition, there exist $\nu,\mu\in T(\alpha)$ with $\nu_1\geq 2$, $\mu_2\geq 2$. Let H be the plane in H_0 through δ,ν, and μ. Then by definition again, there exists a point $\lambda \in T(\alpha)$ not contained in H. Conversely, for fixed $\nu, \mu,$ and λ, we can calculate the weight α. Thus we may classify all the possible triples of points $\{\nu, \mu, \lambda\}$ as above and check the condition $\delta \in \mathrm{Int}\langle T(\alpha)\rangle$. We proceed in 4 steps.

Step 1. We classify points ν in $T(\alpha)$ with $\nu_1 \geq 2$. Since $\alpha_1 \geq \frac{1}{4}$, we have $2 \leq \nu_1 \leq 4$.

Case 1. $\nu_1 = 3$ or 4.

Since $3\alpha_1 + \alpha_j \geq 1$ $(j=1,2,3,4)$, only possible cases are

$\nu = (4,0,0,0), (3,1,0,0), (3,0,1,0)$, $\alpha = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ or

$\nu = (3,0,0,1)$, $\alpha_1 = \alpha_2 = \alpha_3$ or

$\nu = (3,0,0,0)$, $\alpha_1 = \frac{1}{3}$.

Case 3. $\nu_1 = 2$ and $\nu_2 \neq 0$.

Since $2\alpha_1 + \alpha_2 + \alpha_j \geq 1$ $(j=2,3,4)$, we have

$\nu = (2,2,0,0), (2,1,1,0)$, $\alpha = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ or

$\nu = (2,1,0,1)$, $\alpha_1 = \alpha_2 = \alpha_3$ or

$\nu = (2,1,0,0)$, $2\alpha_1 + \alpha_2 = 1$.

Case 4. $\nu_1 = 2$, $\nu_2 = 0$ and $\nu_3 \neq 0$.

Since $2\alpha_1 + \alpha_3 + \alpha_j \geq 1$ $(j=3,4)$,

$\nu = (2,0,2,0)$, $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4$ or

$\nu = (2,0,1,1)$, $\alpha_1 = \alpha_2$ or

$\nu = (2,0,1,0)$, $2\alpha_1 + \alpha_3 = 1$.

Case 5. $\nu=(2,0,0,n)$, $2\alpha_1+n\alpha_4=1$ $(n\geq 0)$.

So if $\alpha\in W_4$, then α satisfies one of the following conditions:

(A) $\alpha_1=\alpha_2$ (i.e., $\nu(2,0,1,1)\in T(\alpha)$).

(B) $\alpha_1=\frac{1}{3}$ (i.e., $\nu(3,0,0,0)\in T(\alpha)$).

(C) $2\alpha_1+\alpha_2=1$ (i.e., $\nu(2,1,0,0)\in T(\alpha)$).

(D) $2\alpha_1+\alpha_3=1$ (i.e., $\nu(2,0,1,0)\in T(\alpha)$).

(E) $2\alpha_1+n\alpha_4=1$ (i.e., $\nu(2,0,0,n)\in T(\alpha)$), $(n\geq 0)$.

Step 2 and 3. Next we classify points $\mu\in T(\alpha)$ with $\mu_2\geq 2$ and determine weights by searching another point λ.

Step 2 of the case (A) $\alpha_1=\alpha_2$ (i.e., $\nu(2,0,1,1)\in T(\alpha)$).

There exists $\mu\in T(\alpha)$ such that $\mu_1=0$ and $2\leq\mu_2\leq 4$. If $\mu_2=2$, then $\nu,\mu,\delta\in\{x_1+x_2=2\}$, so we may assume $3\leq\mu_2$.

(A-1) Assume that $\mu_2=4$.

Then $\alpha_1=\alpha_2=\frac{1}{4}$ and $\alpha=(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$.

(A-2) Assume that $\mu_2=3$.

Since $3\alpha_2+\alpha_j\geq 1$ $(j=3,4)$, we have

$\mu=(0,3,1,0)$ and $\alpha=(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$ or

$\mu=(0,3,0,1)$ or

$\mu=(0,3,0,0)$.

Step 3 of the case (A). If $\mu=(0,3,0,1)$, then $\alpha_1=\alpha_2=\alpha_3$ and

$\nu,\mu,\delta\in\{x_4=1\}$. So there exists $(n,0,0,0)\in T(\alpha)$ with $n\geq 4$. Thus $n=4$ and $\alpha=(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$. If $\mu=(0,3,0,0)$, then $\alpha_1=\alpha_2=\alpha_3+\alpha_4$ and $\nu,\mu,\delta\in\{x_3=x_4\}$. So there exists $(0,0,m,n)\in T(\alpha)$ with $m>n$. Since $(0,0,3,3)\in T(\alpha)$, we have

$\alpha_3=\alpha_4$ and $\alpha=(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6})$ or

$2\alpha_3=3\alpha_4$ and $\alpha=(\frac{1}{3},\frac{1}{3},\frac{1}{5},\frac{2}{15})$ or

$\alpha_3=2\alpha_4$ and $\alpha=(\frac{1}{3},\frac{1}{3},\frac{2}{9},\frac{1}{9})$ or

$\alpha_3=3\alpha_4$ and $\alpha=(\frac{1}{3},\frac{1}{3},\frac{1}{4},\frac{1}{12})$.

Step 2 of the case (B) $\alpha_1=\frac{1}{3}$ (i.e., $\nu(3,0,0,0)\in T(\alpha)$).

Since $\alpha_2+\alpha_3+\alpha_4=\frac{2}{3}$, we have $\frac{2}{9}\leq\alpha_2\leq\frac{1}{3}$, so there exists $\mu\in T(\alpha)$ such that $2\leq\mu_2\leq 4$.

(B-1) Assume $\mu_2=4$.

Since $1=\frac{3}{2}(\alpha_2+\alpha_3+\alpha_4)\leq 4\alpha_2+\frac{1}{2}\alpha_4$,

(B-i) $\mu=(0,4,0,0)$ and $\alpha_1=\frac{1}{3}$, $\alpha_2=\frac{1}{4}$.

(B-2) Assume $\mu_2=3$.

Since $1=\frac{3}{2}(\alpha_2+\alpha_3+\alpha_4)\leq 3\alpha_2+\frac{3}{2}\alpha_4$,

(B-ii) $\mu=(1,3,0,0)$ and $\alpha=(\frac{1}{3},\frac{2}{9},\frac{2}{9},\frac{2}{9})$ or

(B-iii) $\mu=(0,3,1,0)$ and $3\alpha_2+\alpha_3=1$ or

(B-iv) $\mu=(0,3,0,1)$ and $3\alpha_2+\alpha_4=1$ or

(B-v) $\quad \mu=(0,3,0,0)$ and $\alpha_1=\alpha_2=\frac{1}{3}$.

(B-3) Assume $\mu_2=2$.

Since $1=\frac{3}{2}(\alpha_2+\alpha_3+\alpha_4)\leq 2\alpha_2+\alpha_3+\frac{3}{2}\alpha_4$ and $\frac{7}{9}\leq\alpha_1+2\alpha_2\leq 1$,

$\mu=(1,2,1,0)$ and $\alpha=(\frac{1}{3},\frac{2}{9},\frac{2}{9},\frac{2}{9})$ (This is the case (B-ii).)

or

(B-vi) $\quad \mu=(1,2,0,1)$ and $\alpha_2=\alpha_3$ or

$\mu=(1,2,0,0)$ and $\alpha_1=\alpha_2=\frac{1}{3}$ (This is the case (B-v).) or

(B-vii) $\quad \mu=(0,2,2\ 0)$ and $\alpha_2+\alpha_3=\frac{1}{2}$ or

$\mu=(0,2,1,1)$ and $\alpha_1=\alpha_2=\frac{1}{3}$ (This is the case (B-v).) or

(B-viii) $\quad \mu=(0,2,0,n)$ and $2\alpha_2+n\alpha_4=1$ $(n\geq 2)$.

Step 3 of the case (B). We study the above 8 cases in more detail.

Case (B-i). Assume $\alpha_1=\frac{1}{3}$, $\alpha_2=\frac{1}{4}$, $\nu=(3,0,0,0)$ and $\mu=(0,4,0,0)$.

Then $\alpha_3=\frac{5}{12}-\alpha_4$ and $\frac{1}{6}\leq\alpha_4\leq\frac{5}{24}$. If $\lambda\in T(\alpha)$, then

$$\frac{1}{3}\lambda_1+\frac{1}{4}\lambda_2+(\frac{5}{12}-\alpha_4)\lambda_3+\alpha_4\lambda_4=\frac{1}{3}\lambda_1+\frac{1}{4}\lambda_2+\frac{5}{12}\lambda_3+(-\lambda_3+\lambda_4)\alpha_4=1,$$

and since $\nu,\mu,\delta\in\{-x_3+x_4=0\}$, there exists $\lambda\in T(\alpha)$ with $-\lambda_3+\lambda_4<0$.

For this λ, we have

$$\frac{1}{3}\lambda_1+\frac{1}{4}\lambda_2+\frac{5}{24}\lambda_3+\frac{5}{24}\lambda_4\leq 1\leq\frac{1}{3}\lambda_1+\frac{1}{4}\lambda_2+\frac{1}{4}\lambda_3+\frac{1}{6}\lambda_4 \text{ , and thus}$$

$\lambda=(0,0,4,0),(0,1,3,0),(0,2,2,0),(1,0,3,0),(1,0,2,1),(1,1,2,0)$,

and $\alpha=(\frac{1}{3},\frac{1}{4},\frac{1}{4},\frac{1}{6}),(\frac{1}{3},\frac{1}{4},\frac{2}{9},\frac{7}{36}),(\frac{1}{3},\frac{1}{4},\frac{5}{24},\frac{5}{24})$.

Similarly, from the cases (B-iii),$\cdots$,(B-vii), we obtain the following weights.

Case (B-iii). $\alpha=(\frac{1}{3},\frac{4}{15},\frac{1}{5},\frac{1}{5}),(\frac{1}{3},\frac{1}{4},\frac{1}{4},\frac{1}{6}),(\frac{1}{3},\frac{7}{27},\frac{2}{9},\frac{5}{27})$.

Case (B-iv). $\alpha=(\frac{1}{3},\frac{4}{15},\frac{1}{5},\frac{1}{5}),(\frac{1}{3},\frac{7}{24},\frac{1}{4},\frac{1}{8}),(\frac{1}{3},\frac{2}{7},\frac{5}{21},\frac{1}{7}),(\frac{1}{3},\frac{5}{18},\frac{2}{9},\frac{1}{6})$.

Case (B-v). $\alpha=(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}),(\frac{1}{3},\frac{1}{3},\frac{1}{5},\frac{2}{15}),(\frac{1}{3},\frac{1}{3},\frac{2}{9},\frac{1}{9}),(\frac{1}{3},\frac{1}{3},\frac{1}{4},\frac{1}{12})$.

Case (B-vi). $\alpha=(\frac{1}{3},\frac{1}{4},\frac{1}{4},\frac{1}{6}),(\frac{1}{3},\frac{2}{9},\frac{2}{9},\frac{2}{9})$.

case (B-vii). $\alpha=(\frac{1}{3},\frac{1}{4},\frac{1}{4},\frac{1}{6}),(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}),(\frac{1}{3},\frac{5}{18},\frac{2}{9},\frac{1}{6})$.

Case (B-viii). Assume $\alpha_1=\frac{1}{3}$, $2\alpha_2+n\alpha_4=1$ $(n\geq 2)$, $\nu=(3,0,0,0)$, $\mu=(0,2,0,n)$. If $n=2$, then $\alpha_2+\alpha_4=\frac{1}{2}$ and $\alpha=(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6})$. So we consider the case $n>2$. By $\alpha_2=\frac{1}{2}-\frac{n}{2}\alpha_4$ and $\alpha_3=\frac{1}{6}+\frac{n-2}{2}\alpha_4$, we have

$$\frac{1}{3}\lambda_1+\frac{1}{2}\lambda_2+\frac{1}{6}\lambda_3+(-\frac{n}{2}\lambda_2-\frac{n-2}{2}\lambda_3+\lambda_4)\alpha_4=1,$$

for any $\lambda\in T(\alpha)$.

Since $\delta,\nu,\mu\in\{-\frac{n}{2}x_2-\frac{n-2}{2}x_3+x_4=0\}$, there exists $\lambda\in T(\alpha)$ with

$$n\lambda_2>(n-2)\lambda_3+2\lambda_4 \quad \cdots\cdots\cdot (*).$$

By the condition $\alpha_1=\frac{1}{3}$, we may assume $\lambda_1\leq 1$. But in this case, λ_2 is greater than 1, and the case(B-viii) is reduced to the cases (B-i),$\cdots$,(B-vii).

Step 2 of the case (C) $2\alpha_1+\alpha_2=1$ (i.e., $\nu(2,1,0,0)\in T(\alpha)$).

There exists $\mu\in T(\alpha)$ such that $\mu_2\geq 2$. Since $\alpha_2+2\alpha_3+2\alpha_4=1$, we have $\frac{1}{5}\leq\alpha_2$ and $2\leq\mu_2\leq 5$. As in the Step 2 of the case (B), we have the following cases.

(C-i) $\mu=(0,5,0,0),(0,4,1,0)$ etc, and $\alpha=(\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5})$.

(C-ii) $\mu=(0,4,0,0)$ and $\alpha_1=\frac{3}{8}$, $\alpha_2=\frac{1}{4}$.

(C-iii) $\mu=(0,3,0,2),(1,2,0,1),(0,2,1,2)$ and $\alpha_2=\alpha_3$ or

(C-iv) $\mu=(0,3,1,0)$ and $3\alpha_2+\alpha_3=1$ or

(C-v) $\mu=(0,3,0,1)$ and $3\alpha_2+\alpha_4=1$ or

(C-vi) $\mu=(0,3,0,0),(1,2,0,0),(0,2,1,1)$ and $\alpha_1=\alpha_2=\frac{1}{3}$.

(C-vii) $\mu=(0,2,2,0)$ and $\alpha_2+\alpha_3=\frac{1}{2}$ or

(C-viii) $\mu=(0,2,0,n)$ and $2\alpha_2+n\alpha_4=1$.

Step 3 of the case (C). Next, we determine weights α for the above 8 cases. By the same calculation stated in the case (B-i), we have the following weights α from the cases (C-ii),$\cdots$,(C-vii).

Case (C-ii). $\alpha=(\frac{3}{8},\frac{1}{4},\frac{1}{5},\frac{7}{40}),(\frac{3}{8},\frac{1}{4},\frac{5}{24},\frac{1}{6}),(\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{1}{8}),(\frac{3}{8},\frac{1}{4},\frac{3}{16},\frac{3}{16})$.

Case (C-iii). $\alpha=(\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}),(\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{1}{8})$.

Case (C-iv). $\alpha=(\frac{11}{30},\frac{4}{15},\frac{1}{5},\frac{1}{6}),(\frac{7}{19},\frac{5}{19},\frac{4}{19},\frac{3}{19}),(\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{1}{8}),(\frac{4}{11},\frac{3}{11},\frac{2}{11},\frac{2}{11})$.

Case (C-v). $\alpha=(\frac{9}{25},\frac{7}{25},\frac{1}{5},\frac{4}{25}),(\frac{5}{14},\frac{2}{7},\frac{3}{14},\frac{1}{7}),(\frac{7}{20},\frac{3}{10},\frac{1}{4},\frac{1}{10}),$
$(\frac{4}{11},\frac{3}{11},\frac{2}{11},\frac{2}{11}),(\frac{6}{17},\frac{5}{17},\frac{4}{17},\frac{2}{17})$.

Case(C-vi). This case is already appeared in the case (B).

Case (C-vii). $\alpha=(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}),(\frac{7}{20},\frac{3}{10},\frac{1}{5},\frac{3}{20}),(\frac{5}{14},\frac{2}{7},\frac{3}{14},\frac{1}{7}),(\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{1}{8})$.

Case(C-viii). Assume $2\alpha_1+\alpha_2=1$, $2\alpha_2+n\alpha_4=1$ $(n\geq 2)$, $\nu=(2,1,0,0)$, and $\mu=(0,2,0,n)$. If $n=2$, then $\alpha_2+\alpha_4=\frac{1}{2}$ and $\alpha=(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6})$, so we consider the case $n>2$. Since $\alpha_1 = \frac{1}{4} + \frac{n}{4}\alpha_4$, $\alpha_2 = \frac{1}{2} - \frac{1}{2}\alpha_4$,

$\alpha_3 = \frac{1}{4} + \frac{n-4}{4}\alpha_4$, for any $\lambda \in T(\alpha)$

$$\frac{1}{4}\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{4}\lambda_3 + (\frac{n}{4}\lambda_1 - \frac{n}{2}\lambda_2 + \frac{n-4}{4}\lambda_3 + \lambda_4)\alpha_4 = 1.$$

Since $\delta, \nu, \mu \in \{\frac{n}{4}\lambda_1 - \frac{n}{2}\lambda_2 + \frac{n-4}{4}\lambda_3 + \lambda_4 = 0\}$, there exists $\lambda \in T(\alpha)$ with

$$n\lambda_1 - 2n\lambda_2 + (n-4)\lambda_3 + 4\lambda_4 < 0 \quad \cdots\cdots \ (*).$$

By the assumption, we have $2\alpha_1 = \alpha_2 + n\alpha_4$, so we may assume $\lambda_1 \leq 1$.
But in this case, one can easily check that there are no $\lambda \in T(\alpha)$ with $\lambda_2 \leq 1$, or $\alpha = (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}), (\frac{1}{3}, \frac{1}{3}, \frac{2}{9}, \frac{1}{9})$. Thus, case(C-viii) is reduced to the cases (C-i), $\cdots$, (C-vii).

Step 2 of the case (D) $2\alpha_1 + \alpha_3 = 1$ (i.e., $\nu(2,0,1,0) \in T(\alpha)$).

There exists $\mu \in T(\alpha)$ such that $\mu_2 \geq 2$. Since $2\alpha_2 + \alpha_3 + 2\alpha_4 = 1$ and $\alpha_1 < \frac{1}{2}$, we have $\frac{1}{5} \leq \alpha_2 \leq \frac{1}{2}$ and $2 \leq \mu_2 \leq 5$. We may assume $\mu_1 = 0$ for $\alpha_1 = \alpha_2 + \alpha_4$. Then only posibilities are the following 10 cases.

(D-i) $\mu = (0,5,0,0), (0,4,1,0)$ etc, and $\alpha = (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$.

(D-ii) $\mu = (0,4,0,0)$ and $\alpha_2 = \frac{1}{4}$.

(D-iii) $\mu = (0,3,0,2)$ and $\alpha_2 = \alpha_3$ or

(D-iv) $\mu = (0,3,1,0)$ and $3\alpha_2 + \alpha_3 = 1$ or

(D-v) $\mu = (0,3,0,1)$ and $3\alpha_2 + \alpha_4 = 1$ or

(D-vi) $\mu = (0,3,0,0)$ and $\alpha_2 = \frac{1}{3}$.

(D-vii) $\mu=(0,2,3,0),(0,2,2,1),(0,2,0,3)$ and $\alpha_3=\alpha_4$ or

(D-viii) $\mu=(0,2,1,2)$ (This μ is always in $T(\alpha)$.) or

(D-ix) $\mu=(0,2,2,0),(0,2,0,4)$ and $\alpha_3=2\alpha_4$ or

(D-x) $\mu=(0,2,0,n)$ $(n\geq 3)$ and $2\alpha_2+n\alpha_4=1$.

Step 3 of the case (D). Each cases are investigated more precisely in the followings. From the cases (D-ii),$\cdots$,(D-vi), we obtain the following weights.

Case(D-ii). $\alpha=(\frac{5}{12},\frac{1}{4},\frac{1}{6},\frac{1}{6}),(\frac{2}{5},\frac{1}{4},\frac{1}{5},\frac{3}{20}),(\frac{11}{28},\frac{1}{4},\frac{3}{14},\frac{1}{7}),(\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{1}{8}),$

$(\frac{13}{32},\frac{1}{4},\frac{3}{16},\frac{5}{32})$.

Case(D-iii). Assume $2\alpha_1+\alpha_3=1$, $\alpha_2=\alpha_3$, $\nu=(2,0,1,0)$, and $\mu=(0,3,0,2)$. Then $(2,1,0,0)$ is contained in $T(\alpha)$, and this case is considered in (C).

Case (D-iv). $\alpha=(\frac{3}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7}),(\frac{5}{12},\frac{5}{18},\frac{1}{6},\frac{5}{36}),(\frac{12}{29},\frac{8}{29},\frac{5}{29},\frac{4}{29}),(\frac{9}{22},\frac{3}{11},\frac{2}{11},\frac{3}{22}),$

$(\frac{2}{5},\frac{4}{15},\frac{1}{5},\frac{2}{15}),(\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{1}{8}),(\frac{9}{23},\frac{6}{23},\frac{5}{23},\frac{3}{23})$.

Case (D-v). $\alpha=(\frac{3}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7}),(\frac{5}{12},\frac{7}{24},\frac{1}{6},\frac{1}{8}),(\frac{7}{17},\frac{5}{17},\frac{3}{17},\frac{2}{17}),(\frac{2}{5},\frac{3}{10},\frac{1}{5},\frac{1}{10}),$

$(\frac{5}{13},\frac{4}{13},\frac{3}{13},\frac{1}{13}),(\frac{3}{8},\frac{5}{16},\frac{1}{4},\frac{1}{16})$.

Case (D-vi). $\alpha=(\frac{4}{9},\frac{1}{3},\frac{1}{9},\frac{1}{9}),(\frac{21}{48},\frac{1}{3},\frac{1}{8},\frac{5}{48}),(\frac{17}{39},\frac{1}{3},\frac{5}{39},\frac{4}{39}),(\frac{13}{30},\frac{1}{3},\frac{2}{15},\frac{1}{10}),$

$(\frac{3}{7},\frac{1}{3},\frac{1}{7},\frac{2}{21})$ $(\frac{5}{12},\frac{1}{3},\frac{1}{6},\frac{1}{12}),(\frac{14}{33},\frac{1}{3},\frac{5}{33},\frac{1}{11}),(\frac{2}{5},\frac{1}{3},\frac{1}{5},\frac{1}{15}),$

$(\frac{11}{27},\frac{1}{3},\frac{5}{27},\frac{2}{27}),(\frac{7}{18},\frac{1}{3},\frac{2}{9},\frac{1}{18}),(\frac{8}{21},\frac{1}{3},\frac{5}{21},\frac{1}{21}),(\frac{3}{8},\frac{1}{3},\frac{1}{4},\frac{1}{24})$.

Case(D-vii) and Case(D-ix) will be studied in Case(D-x), for they are the special cases of Case(D-x).

Case(D-viii). Since $\delta,\nu,\mu\in\{x_1+3x_2-2x_3-2x_4=0\}=H$,

and $\{(2,0,1,0),(1,1,1,1),(0,2,1,2)\}\subset H\cap T(\alpha)$, there exists $\lambda\in T(\alpha)$ such that $\lambda=(2,1,0,0)$ or $\lambda_2\geq 2$, $\lambda\neq\mu$.

Case (D-x). Assume $2\alpha_1+\alpha_3=1$, $2\alpha_2+n\alpha_4=1$ $(n\geq 3)$, $\nu=(2,0,1,0)$, and $\mu=(0,2,0,n)$. Then $\alpha_1=\frac{1}{2}-\frac{n-2}{2}\alpha_4$, $\alpha_2=\frac{1}{2}-\frac{n}{2}\alpha_4$, $\alpha_3=(n-2)\alpha_4$, and for any $\lambda\in T(\alpha)$

$$\frac{1}{2}\lambda_1+\frac{1}{2}\lambda_2+\{-\frac{n-2}{2}\lambda_1-\frac{n}{2}\lambda_2+(n-2)\lambda_3+\lambda_4\}\alpha_4=1.$$

Since $\delta,\nu,\mu\in\{-\frac{n-2}{2}x_1-\frac{n}{2}x_2+(n-2)x_3+x_4=0\}=H$, there exists $\lambda\in T(\alpha)$ with

$$2(n-2)\lambda_3+2\lambda_4<(n-2)\lambda_1+n\lambda_2 .$$

Then, $H\cap T(\alpha)=\{(2,0,0,n-2),(2,0,1,0),(1,1,0,n-1),(1,1,1,1),$

Then, $H\cap T(\alpha)=\{(2,0,0,n-2),(2,0,1,0),(1,1,0,n-1),(1,1,1,1),$

$(0,2,0,n),(0,2,1,2),\cdots\}$

and thus $\lambda=(2,0,1,0)$ or $\lambda_2\geq 2$. So the cases (D-vii),(D-ix),(D-x) are reduced to the cases (D-i),$\cdots$,(D-vi).

(E) $2\alpha_1+n\alpha_4=1$ $(n\geq 0)$. (i.e., $\nu(2,0,0,n)\in T(\alpha)$.)

(E-0) First we consider the case $n=0$.

Step 2 of the case (E-0). Since $\alpha_1=\alpha_2+\alpha_3+\alpha_4=\frac{1}{2}$, there exists $\mu\in T(\alpha)$ with $\mu_1=0$, $3\leq\mu_2\leq 6$. Thus we have the following cases.

(E-0-i) $\mu=(0,0,6,0),(0,5,1,0)$ etc, and $\alpha=(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6})$.

(E-0-ii) $\mu=(0,5,0,0)$ and $\alpha=\frac{1}{5}$.

(E-0-iii) $\mu=(0,4,0,2),(0,3,1,2)$ and $\alpha_2=\alpha_3$ or

(E-0-iv) $\mu=(0,4,1,0)$ and $4\alpha_2+\alpha_3=1$ or

(E-0-v) $\mu=(0,4,0,1)$ and $4\alpha_2+\alpha_4=1$ or

(E-0-vi) $\mu=(0,4,0,0),(0,3,1,1)$ and $\alpha_2=\frac{1}{4}$.

(E-0-vii) $\mu=(0,3,2,0)$ and $\alpha_2=2\alpha_4$ or

(E-0-viii) $\mu=(0,3,1,0)$ and $3\alpha_2+\alpha_3=1$ or

(E-0-ix) $\mu=(0,3,0,0)$ and $\alpha_2=\frac{1}{3}$ or

(E-0-x) $\mu=(0,3,0,n)$ $(n\geq 1)$ and $3\alpha_2+n\alpha_4=1$.

Step 3 of the case (E-0). We study the above 10 cases in more detail. From the cases (E-0-ii),$\cdots$,(E-0-ix), we obtain the following weights.

Case(E-0-ii). $\alpha=(\frac{1}{2},\frac{1}{5},\frac{1}{6},\frac{2}{15}),(\frac{1}{2},\frac{1}{5},\frac{7}{40},\frac{1}{8}),(\frac{1}{2},\frac{1}{5},\frac{1}{5},\frac{1}{10}),(\frac{1}{2},\frac{1}{5},\frac{4}{25},\frac{7}{50}),$

$(\frac{1}{2},\frac{1}{5},\frac{3}{20},\frac{3}{20})$.

Case(E-0-iii). $\alpha=(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}),(\frac{1}{2},\frac{1}{5},\frac{1}{5},\frac{1}{10})$.

Case(E-0-iv). $\alpha=(\frac{1}{2},\frac{3}{14},\frac{1}{7},\frac{1}{7}),(\frac{1}{2},\frac{5}{24},\frac{1}{6},\frac{1}{8}),(\frac{1}{2},\frac{7}{34},\frac{3}{17},\frac{2}{17}),(\frac{1}{2},\frac{1}{5},\frac{1}{5},\frac{1}{10}),$

$(\frac{1}{2},\frac{4}{19},\frac{3}{19},\frac{5}{38})$.

Case(E-0-v). $\alpha=(\frac{1}{2},\frac{3}{14},\frac{1}{7},\frac{1}{7}),(\frac{1}{2},\frac{2}{9},\frac{1}{6},\frac{1}{9}),(\frac{1}{2},\frac{5}{22},\frac{2}{11},\frac{1}{11}),(\frac{1}{2},\frac{7}{30},\frac{1}{5},\frac{1}{15}),$

$(\frac{1}{2},\frac{7}{32},\frac{5}{32},\frac{1}{8}),(\frac{1}{2},\frac{3}{13},\frac{5}{26},\frac{1}{13})$.

Case(E-0-vi). $\alpha=(\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{8}),(\frac{1}{2},\frac{1}{4},\frac{1}{7},\frac{3}{28}),(\frac{1}{2},\frac{1}{4},\frac{3}{20},\frac{1}{10}),(\frac{1}{2},\frac{1}{4},\frac{1}{6},\frac{1}{12}),$

$(\frac{1}{2},\frac{1}{4},\frac{3}{16},\frac{1}{16}),(\frac{1}{2},\frac{1}{4},\frac{1}{5},\frac{1}{20})$.

Case(E-0-vii). $\alpha=(\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{8}),(\frac{1}{2},\frac{5}{21},\frac{1}{7},\frac{5}{42}),(\frac{1}{2},\frac{4}{17},\frac{5}{34},\frac{2}{17}),(\frac{1}{2},\frac{3}{13},\frac{2}{13},\frac{3}{26}),$

$(\frac{1}{2},\frac{2}{9},\frac{1}{6},\frac{1}{9}),(\frac{1}{2},\frac{1}{5},\frac{1}{5},\frac{1}{10}),(\frac{1}{2},\frac{3}{14},\frac{5}{28},\frac{3}{28}).$

Case(E-0-viii). $\alpha=(\frac{1}{2},\frac{3}{10},\frac{1}{10},\frac{1}{10}),(\frac{1}{2},\frac{8}{27},\frac{1}{9},\frac{4}{54}),(\frac{1}{2},\frac{13}{44},\frac{5}{44},\frac{1}{11}),(\frac{1}{2},\frac{5}{17},\frac{2}{17},\frac{3}{34}),$

$(\frac{1}{2},\frac{7}{24},\frac{1}{8},\frac{1}{12}),(\frac{1}{2},\frac{2}{7},\frac{1}{7},\frac{1}{14}),(\frac{1}{2},\frac{11}{38},\frac{5}{38},\frac{3}{38}),(\frac{1}{2},\frac{3}{11},\frac{2}{11},\frac{1}{22}),$

$(\frac{1}{2},\frac{5}{18},\frac{1}{6},\frac{1}{18}),(\frac{1}{2},\frac{7}{26},\frac{5}{26},\frac{1}{26}),(\frac{1}{2},\frac{4}{15},\frac{1}{5},\frac{1}{30}).$

Case(E-0-ix). $\alpha=(\frac{1}{2},\frac{1}{4},\frac{1}{12},\frac{1}{12}),(\frac{1}{2},\frac{1}{3},\frac{1}{11},\frac{5}{66}),(\frac{1}{2},\frac{1}{3},\frac{5}{54},\frac{2}{27}),(\frac{1}{2},\frac{1}{3},\frac{2}{21},\frac{1}{14}),$

$(\frac{1}{2},\frac{1}{3},\frac{1}{10},\frac{1}{15}),(\frac{1}{2},\frac{1}{3},\frac{1}{9},\frac{1}{18}),(\frac{1}{2},\frac{1}{3},\frac{5}{48},\frac{1}{16}),(\frac{1}{2},\frac{1}{3},\frac{1}{8},\frac{1}{24}),$

$(\frac{1}{2},\frac{1}{3},\frac{5}{42},\frac{1}{21}),(\frac{1}{2},\frac{1}{3},\frac{2}{15},\frac{1}{30}),(\frac{1}{2},\frac{1}{3},\frac{5}{36},\frac{1}{36}),(\frac{1}{2},\frac{1}{3},\frac{1}{7},\frac{1}{42}).$

Case(E-0-x). Assume $\alpha_1=\frac{1}{2}$, $3\alpha_2+n\alpha_4= 1$, $\nu=(2,0,0,0)$, and $\mu=(0,3,0,n)$. Then $\alpha_2=\frac{1}{3}-\frac{n}{3}\alpha_4$, $\alpha_3=\frac{1}{6}+\frac{n-3}{3}\alpha_4$, and for any $\lambda\in T(\alpha)$,

$$\frac{1}{2}\lambda_1+\frac{1}{3}\lambda_2+\frac{1}{6}\lambda_3+(-\frac{n}{3}\lambda_2+\frac{n-3}{3}\lambda_3+\lambda_4)\alpha_4 = 1.$$

First, assume $n=1$. Then we have

$$\alpha=(\frac{1}{2},\frac{3}{10},\frac{1}{10},\frac{1}{10}),(\frac{1}{2},\frac{11}{36},\frac{1}{9},\frac{1}{12}),(\frac{1}{2},\frac{4}{13},\frac{3}{26},\frac{1}{13}),(\frac{1}{2},\frac{5}{16},\frac{1}{8},\frac{1}{16}),$$

$$(\frac{1}{2},\frac{7}{22},\frac{3}{22},\frac{1}{22}),(\frac{1}{2},\frac{9}{28},\frac{1}{7},\frac{1}{28}).$$

Next, we assume $n\geq 2$. Let $H=\{-\frac{n}{3}x_1+\frac{n-3}{3}x_2+x_4= 0\}$.

Since $\delta,\nu,\mu\in H$, and $(0,3,0,n),(0,2,2,2)\in H\cap\{x_1= 0\}$, there exists $\lambda\in T(\alpha)$ with $(n-3)\lambda_3+3\lambda_4< n\lambda_2$, $\lambda_2\geq 3$, and thus Case(E-0-x) ($n\geq 2$) is reduced to the cases (E-0-i),$\cdots$(E-0-ix).

(E-1) Next, we assume $n=1$.

Step 2 of the case (E-1). Since $2\alpha_2+2\alpha_3+2\alpha_4=1$, there exists $\mu\in T(\alpha)$ with $2\leq\mu_2\leq 5$. The possibilities are the 9 cases below.

(E-1-i) $\mu=(0,0,5,0),(0,4,1,0)$ etc, and $\alpha=(\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5})$.

(E-1-ii) $\mu=(0,4,0,1),(0,3,1,1),(1,2,0,1)$ and $\alpha_2=\alpha_3$ or

(E-1-iii) $\mu=(0,4,0,0)$ and $\alpha_2=\frac{1}{4}$.

(E-1-iv) $\mu=(0,3,1,0),(1,2,0,0)$ and $3\alpha_2+\alpha_3=1$ or

(E-1-v) $\mu=(0,3,0,n)$ $(n\geq 0)$ and $3\alpha_2+n\alpha_4=1$ or

(E-1-vi) $\mu=(0,2,3,0)$ and $\alpha_3=\alpha_4$ or

(E-1-vii) $\mu=(0,2,2,1)$ (This μ is always in $T(\alpha)$.) or

(E-1-viii) $\mu=(0,2,1,n)$ $(n\geq 2)$ and $2\alpha_2+\alpha_3+n\alpha_4=1$ or

(E-1-ix) $\mu=(0,2,0,n)$ $(m\geq 3)$ and $2\alpha_2+n\alpha_4=1$.

Step 3 of the case (E-1). We will study the above 9 cases in more detail.

Case(E-1-ii). Assume $2\alpha_1+\alpha_4=1$, $\alpha_2=\alpha_3$, $\nu=(2,0,0,1)$, and $\mu=(0,4,0,1)$. Then $\alpha_1=\frac{1}{2}-\frac{1}{2}\alpha_4$, $\alpha_2=\alpha_3=\frac{1}{4}-\frac{1}{4}\alpha_4$, $0<\alpha_4\leq\frac{1}{5}$, and for any $\lambda\in T(\alpha)$,

$$\frac{1}{2}\lambda_1+\frac{1}{4}\lambda_2+\frac{1}{4}\lambda_3+(-\frac{1}{2}\lambda_1-\frac{1}{4}\lambda_2-\frac{1}{4}\lambda_3+\lambda_4)\alpha_4=1.$$

Since $\delta,\nu,\mu\in\{-\frac{1}{2}\lambda_1-\frac{1}{4}\lambda_2-\frac{1}{4}\lambda_3+\lambda_4=0\}=H$, there exists $\lambda\in T(\alpha)$ with

$$-\frac{1}{2}\lambda_1-\frac{1}{4}\lambda_2-\frac{1}{4}\lambda_3+\lambda_4<0.$$

But one can easily check that $H\cap T(\alpha)\subset\{x_4=1\}$, and thus

$$\alpha=(\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}).$$

Case(E-1-iii). By the same procedure as in the case (B-i), we have

$$\alpha=(\frac{5}{12},\frac{1}{4},\frac{1}{6},\frac{1}{6}),(\frac{9}{20},\frac{1}{4},\frac{1}{5},\frac{1}{10}),(\frac{7}{16},\frac{1}{4},\frac{3}{16},\frac{1}{8}).$$

Case(E-1-iv). Similarly,

$$\alpha=(\frac{3}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7}),(\frac{4}{9},\frac{5}{18},\frac{1}{6},\frac{1}{9}),(\frac{5}{11},\frac{3}{11},\frac{2}{11},\frac{1}{11}),(\frac{7}{15},\frac{4}{15},\frac{1}{5},\frac{1}{15}).$$

Case(E-1-v). Assume $2\alpha_1+\alpha_4=1$, $3\alpha_2+n\alpha_4=1$, $\nu=(2,0,0,1)$, and $\mu=(0,3,0,n)$. Then $\alpha_1=\frac{1}{2}-\frac{1}{2}\alpha_4$, $\alpha_2=\frac{1}{3}-\frac{n}{3}\alpha_4$, $\alpha_3=\frac{1}{6}+\frac{2n-3}{6}\alpha_4$, and for any $\lambda\in T(\alpha)$,

$$\frac{1}{2}\lambda_1+\frac{1}{3}\lambda_2+\frac{1}{6}\lambda_3+(-\frac{1}{2}\lambda_1-\frac{n}{3}\lambda_2+\frac{2n-3}{6}\lambda_3+\lambda_4)\alpha_4=1.$$

Since $\delta,\nu,\mu\in\{-\frac{1}{2}\lambda_1-\frac{n}{3}\lambda_2+\frac{2n-3}{6}\lambda_3+\lambda_4=0\}$, there exists $\lambda\in T(\alpha)$ with

$$(2n-3)\lambda_3+6\lambda_4<3\lambda_1+2n\lambda_2$$

First, we consider the case $n=0$. Then we have

$$\alpha=(\frac{4}{9},\frac{1}{3},\frac{1}{9},\frac{1}{9}),(\frac{11}{24},\frac{1}{3},\frac{1}{8},\frac{1}{12}),(\frac{7}{15},\frac{1}{3},\frac{2}{15},\frac{1}{15}),(\frac{10}{21},\frac{1}{3},\frac{1}{7},\frac{1}{21}).$$

Next, assume $n=1$. Then $H\cap T(\alpha)\subset\{x_4=1\}$, and thus

$$\alpha=(\frac{3}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7}).$$

And if $n\geq 2$, then one can find $\lambda\in T(\alpha)$ such that $\lambda_2\geq 2$ and $\lambda\neq(0,3,0,n),(0,2,0,m)$ $(m\geq 3)$, $(0,2,2,1)$, and this case is reduced to the cases (E-1-i),$\cdots$,(E-1-iv).

Case(E-1-vi). In this case, the point $(2,0,1,0)$ is in $T(\alpha)$, so we already considered this case in (D).

Case(E-1-vii). In this case, one can easily check that there exists another $\mu\in T(\alpha)$ with $\mu_2\geq 2$.

Case(E-1-viii). Assume $2\alpha_1+\alpha_4=1$, $2\alpha_2+\alpha_3+n\alpha_4=1$ $(n\geq 2)$. Then $\alpha_3=(n-1)\alpha_4$ and $(0,2,0,2n-1)\in T(\alpha)$, so this case is reduced to the next case (E-1-ix).

Case(E-1-ix). Assume $2\alpha_1+\alpha_4=1$, $2\alpha_2+n\alpha_4=1$ $(n\geq 3)$.

Then $\alpha_1=\frac{1}{2}-\frac{1}{2}\alpha_4$, $\alpha_2=\frac{1}{2}-\frac{n}{2}\alpha_4$, and $\alpha_3=\frac{n-1}{2}\alpha_4$.

Let $H=\{-\frac{1}{2}x_1-\frac{n}{2}x_2+\frac{n-1}{2}x_3+x_4=0\}$. Then $\delta,\nu,\mu\in H$ and

$$H\cap T(\alpha)\cap\{x_1=0\}\subset\{x_2=2\},\ H\cap T(\alpha)\cap\{x_1=1\}\subset\{x_2=1\}.$$

Thus, if $\lambda\in\{-\frac{1}{2}x_1-\frac{n}{2}x_2+\frac{n-1}{2}x_3+x_4<0\}\cap T(\alpha)$, then $\lambda_2\geq 2$, and hence this case is reduced to the cases (E-1-i),$\cdots$,(E-1-v).

(E-2) Let $n\geq 2$.

Step 2 of the case (E-2). There exists $\mu\in T(\alpha)$ such that $\mu_1>\mu_4$.

(This condition for μ is different from the above cases.)

By cases (A),$\cdots$,(D), we may assume that $\mu_1=1$ and $\mu_4=0$.

(E-2-i) $\mu=(1,3,0,0),(1,2,1,0)$ and $\alpha_2=\alpha_3=\alpha_4$.

(E-2-ii) $\mu=(1,1,2,0)$ and $\alpha_3=\alpha_4$.

(E-2-iii) $\mu=(1,2,0,0)$ and $\alpha_1+2\alpha_2=1$.

(E-2-iv) $\mu=(1,0,m,0)$ $(m\geq 3)$ and $\alpha_1+m\alpha_3=1$.

Step 3 of the case (E-2). We determine weights for the above 4 cases.

Case(E-2-i) and (E-2-ii). In these cases, $(2,0,n,0)\in T(\alpha)$.

Then $n=2$, and $\alpha_1=\alpha_2$. This is the case (A).

Case(E-2-iii). Assume $2\alpha_1+n\alpha_4=1$ $(n\geq 2)$, $\alpha_1+2\alpha_2=1$, $\nu=(2,0,0,n)$, and $\mu=(1,2,0,0)$. Then $\alpha_1=\frac{1}{2}-\frac{n}{2}\alpha_4$, $\alpha_2=\frac{1}{4}+\frac{n}{4}\alpha_4$, $\alpha_3=\frac{1}{4}+\frac{n-4}{4}\alpha_4$, and for any $\lambda\in T(\alpha)$,

$$\frac{1}{2}\lambda_1+\frac{1}{4}\lambda_2+\frac{1}{4}\lambda_3+(-\frac{n}{2}\lambda_1+\frac{n}{4}\lambda_2+\frac{n-4}{4}\lambda_3+\lambda_4)\alpha_4=1.$$

Since $\delta,\nu,\mu\in\{-\frac{n}{2}x_1+\frac{n}{4}x_2+\frac{n-4}{4}x_3+x_4=0\}=H$, there exists $\lambda\in T(\alpha)$ with

$$n\lambda_2+(n-4)\lambda_3+4\lambda_4<2n\lambda_1 .$$

If $n=2$, then $\{(0,2,2,0),(0,1,3,1),(0,0,4,2)\}=H\cap T(\alpha)\cap\{x_1=0\}$ and $\{(1,2,0,0),(1,1,1,1),(1,0,2,2)\}=H\cap T(\alpha)\cap\{x_1=1\}$. So if $\lambda_1\leq 1$, then $\alpha_3=\alpha_4$ or $\alpha_3=2\alpha_4$, and thus

$$\alpha=(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}),(\frac{2}{5},\frac{3}{10},\frac{1}{5},\frac{1}{10}).$$

If $n\geq 3$, then $H\cap T(\alpha)\cap\{x_1=0\}=\{(1,0,4,1)\}$ or ϕ, and $\{(1,2,0,0),(1,1,1,1),(1,0,2,2)\}=H\cap T(\alpha)\cap\{x_1=1\}$. So $\lambda_1\geq 2$, and this case is reduced to the cases (A),$\cdots$,(D).

Case(E-2-iv). Assume $2\alpha_1+n\alpha_4=1$ $(n\geq 2)$, $\alpha_1+m\alpha_3=1$ $(m\geq 3)$, $\nu=(2,0,0,n)$, and $\mu=(1,0,m,0)$. Then $\alpha_1=\frac{1}{2}-\frac{n}{2}\alpha_4$, $\alpha_2=\frac{m-1}{2m}+\frac{mn-n-2m}{2m}\alpha_4$, $\alpha_3=\frac{1}{2m}+\frac{n}{2m}\alpha_4$, and for any $\lambda\in T(\alpha)$

$$\frac{1}{2}\lambda_1+\frac{m-1}{2m}\lambda_2+\frac{1}{2m}\lambda_3+(-\frac{n}{2}\lambda_1+\frac{mn-n-2m}{2m}\lambda_2+\frac{n}{2m}\lambda_3+\lambda_4)\alpha_4=1.$$

Since $\delta,\nu,\mu\in\{-\frac{n}{2}x_1+\frac{mn-n-2m}{2m}x_2+\frac{n}{2m}x_3+x_4=0\}=H$, there exists $\lambda\in T(\alpha)$ with

$$(mn-n-2m)\lambda_2+n\lambda_3+2m\lambda_4<mn\lambda_1 .$$

We can assume that $\{\nu\}=T(\alpha)\cap\{x_2\geq 2\}$, if not, this case is reduced to the cases (A),$\cdots$,(D). In particular, we obtain $\alpha_3 > \alpha_4$. But then, one can easily check that $\lambda_1 = 1$, $\lambda_3 = 0$, and $\lambda_2 \geq 2$. Hence $\lambda = (1,2,0,0)$ and this case is reduced to the case (E-2-iii).

Step 4. In conclusion, we check the condition $(1,1,1,1) \in \mathrm{Int}\langle T(\alpha)\rangle$ for α's obtained by the calculation in (A),$\cdots$,(E), and we obtain 95 weights listed in Table 2. 2.

Q.E.D.

Remark 2.4. The set of weights W_4 coinsides with the set A_4' in M. Reid[10], while the inclusion $A_4' \subset W_4$ is clear. For the list of A_4', see [1].

§3. Minimal resolution

Here we study the minimal resolution of a hypersurface simple K3 singularity (X,x) which is defined by a non-degenerate polynomial f. The next theorem was proved by M. Tomari[12],[13] in terms of the filtered rings without the assumption that f is non-degenerate.

Theorem 3.1(M. Tomari[12],[13]). Let (X,x) be as above (here, we assume that f is non-degenerate with $\alpha(f)=(\frac{p_1}{p},\frac{p_2}{p},\frac{p_3}{p},\frac{p_4}{p})$). If $\pi:(\tilde{X},E) \longrightarrow (X,x)$ is the filtered blow-up with weight (p_1,p_2,p_3,p_4) , then π is a minimal resolution of (X,x).

Moreover, if f is semi quasi-homogeneous, i.e., $\{f_{\Delta_0} = 0\}$ has the isolated singularity (which is also simple K3 singularity) at the origin $0\in\mathbb{C}^4$, then π is the only minimal resolution for (X,x).

To investigate the above filtered blow-up π, we begin with the filtered blow-up of $\mathbb{C}^4$ at the origin $0\in\mathbb{C}^4$. Let $p=(p_1,p_2,p_3,p_4)$ be the 4-ple of positive integers with g.c.d.$(p_i,p_j,p_k)= 1$ for all distinct i,j,k. Then the filtered blow-up of $\mathbb{C}^4$ with weight p,

$$\Pi:(V,F) \longrightarrow (\mathbb{C}^4,0) ,$$

is constructed as follows by using the method of torus embeddings.

First we introduce the notations (c.f.[6],[8]). Let $N:=\mathbb{Z}^4$ and $M:=\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ be the dual $\mathbb{Z}$-module of N. A subset σ of $N_{\mathbb{R}}=N\otimes_{\mathbb{Z}}\mathbb{R}$ is called a cone if there exists $n_1,\cdots,n_s\in N$ such that σ is written by $\sigma = \{\sum_{i=1}^{s} t_i n_i \mid t_i \in \mathbb{R}_0\}$, in this case, we write $\sigma = \langle n_1, \cdots, n_s \rangle$ and call $n_1, \cdots, n_s$ the generators of σ. For a cone σ in $N_{\mathbb{R}}$, we define the dual cone of σ by $\check{\sigma} = \{m \in M_{\mathbb{R}} \mid m(u) \geq 0$ for any $u \in \sigma\}$, and associate a normal variety X_σ with the cone σ:

$$\sigma \rightsquigarrow X_\sigma = \mathrm{Spec}\,\mathbb{C}[M \cap \check{\sigma}]$$

where $\mathbb{C}[M \cap \check{\sigma}]$ is a $\mathbb{C}$-algebra generated by z^m $(m \in M \cap \check{\sigma})$.

Remark 3.2. In this paper, we assume that the generators $n_1, \cdots, n_s$ of a cone σ consist of primitive elements of N, i.e., each n_i satisfies $n_i\mathbb{R} \cap N = n_i\mathbb{Z}$. We define a determinant of a cone σ, $\det \sigma$, by the greatest common divisor of all (s,s) minors of the matrix (n_{ij}), where $n_i = (n_{i1}, \cdots, n_{i4})$. (c.f.[9].)

Let $\sigma \subset N_{\mathbb{R}} = \mathbb{R}^4$ be the first quadrant of $\mathbb{R}^4$, i.e., $\sigma = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$ where $\mathbf{e}_1 = (1,0,0,0), \cdots, \mathbf{e}_4 = (0,0,0,1)$. We divide the cone σ into 4 cones by adding the point $p = (p_1, p_2, p_3, p_4)$ in σ.

$$\sigma = \bigcup_{i=1}^{4} \sigma_i, \qquad \sigma_i = \langle p, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_\ell \rangle.$$

By the inclusions $\sigma_i \subset \sigma$, natural morphisms

$$\pi_i : V_i = \mathrm{Spec}\,\mathbb{C}[\check{\sigma}_i \cap M] \longrightarrow \mathrm{Spec}\,\mathbb{C}[\check{\sigma} \cap M] = \mathbb{C}^4$$

are obtained. Let V be the union of $V_i (i=1,2,3,4)$ which is glued along the images of Π_i. Then we have a morphism

$$\Pi : V \longrightarrow \mathbb{C}^4$$

and one can easily check that $V - \Pi^{-1}(0) \simeq \mathbb{C}^4 - \{0\}$ and $\Pi^{-1}(0)$ is the weighted projective space $\mathbb{P}(p_1,p_2,p_3,p_4)$. We set

$$F := \Pi^{-1}(0) = \mathbb{P}(p_1,p_2,p_3,p_4).$$

Remark 3.3. The normal variety V is a torus embedding associated to the fan $\Gamma^*(f)$ which is called the dual Newton boundary of f in [9], and denoted by $V = T_N\mathrm{emb}(\Gamma^*(f))$. (See [6], [8].)

The filtered blow-up of (X,x) is obtained by means of the above morphism Π. Write $\alpha(f) = (\frac{p_1}{p}, \frac{p_2}{p}, \frac{p_3}{p}, \frac{p_4}{p})$ as in §1 and construct Π for the weight $p=(p_1,p_2,p_3,p_4)$. Let $\tilde{X}$ be the proper transform of X by Π, and set $\pi = \Pi|_{\tilde{X}}$, $E = \pi^{-1}(0)$. Then $\pi : (X,E) \longrightarrow (X,x)$ is the filtered blow-up with weight p.

Next we study the structure of V for a general weight $p=(p_1,p_2,p_3,p_4)$ with $\text{g.c.d.}(p_i,p_j,p_k) = 1$ for all distinct i,j,k. Let $a_{ij} = \text{g.c.d.}(p_i,p_j)$ and set $z_{ij} = z_i^{-(p_j/a_{ij})} \cdot z_j^{(p_i/a_{ij})}$. We can take $z_i, z_{ij}, z_{ik}, z_{i\ell}$ as a parameter system on V_i.

$$\tau_{ij} = < m_j', \ -p_k \cdot m_j' + a_{ij} \cdot m_k', \ -p_\ell \cdot m_j' + a_{ij} \cdot m_\ell' >.$$

If we use the base $\{e_j', e_k', e_\ell'\}$ of $N' = \mathrm{Hom}(M', \mathbb{Z})$ such that $e_s'(m_t') = \delta_{s,t}$, then by this coordinate, τ_{ij} is written by

$$\tau_{ij} = <(0,1,0), (0,0,1), (a_{ij}, p_k, p_\ell)>.$$

(2) We have $\{z_{ij} = 0\}$ in $U_i \simeq$

$$\mathrm{Spec}\mathbb{C}\left[< m_i \ , \ -\frac{p_k}{a_{ik}} \cdot m_i + \frac{p_i}{a_{ik}} \cdot m_k \ , \ -\frac{p_\ell}{a_{i\ell}} \cdot m_i + \frac{p_i}{a_{i\ell}} \cdot m_\ell > \cap \mathbb{Z}^3 \right].$$

So the assertion is obvious.

Corollary 3.5. If $(\frac{p_1}{p}, \frac{p_2}{p}, \frac{p_3}{p}, \frac{p_4}{p}) \in W_4$, then

(1) $\mathrm{Spec}\mathbb{C}[\check{\tau}_{ij} \cap \mathbb{Z}^3]$ has a terminal singularity at worst, and

(2) If $p_i | (p - p_j)$, then $\mathrm{Spec}\mathbb{C}[\check{\rho}_{ij} \cap \mathbb{Z}^3]$ has a terminal singularity at worst.

The above corollary follows directly from Lemma 3.6. below. First we recall the notion of cyclic quotient singularities for the case of dimension 3. Let ξ be a primitive n-th root of unity 1, and let p, q be integers with g.c.d.$(n,p,q) = 1$. We define a equivalent relation on $\mathbb{C}^3$ by $(x_1, x_2, x_3) \sim (\xi x_1, \xi^p x_2, \xi^q x_3)$. Then $X = \mathbb{C}^3/\sim$ is expressed in terms of torus embeddings as

$$X = \mathrm{Spec}\mathbb{C}[\check{\sigma} \cap \mathbb{Z}^3], \text{ where } \sigma = <(n,-p,-q), (0,1,0), (0,0,1)>.$$

In particular, X has an isolated singularity if and only if g.c.d.(n,p)=g.c.d.$(n,q) = 1$.

Lemma 3.6(Terminal lemma[5],[7]). In the above situation, X has a terminal singularity if and only if X has an isolated singularity and one of the following conditions are satisfied;

(1) $-p \equiv 1 \pmod{n}$ or

(2) $-q \equiv 1 \pmod{n}$ or

(3) $p+q \equiv 0 \pmod{n}$.

Next, we prepare some facts about the weighted projective space $\mathbb{P}(p_1,p_2,p_3,p_4)$. Let a_i be an integer defined by $p_i=a_i a_{ij} a_{ik} a_{i\ell}$.

Proposition 3.7.

(1) There exist a cone σ_{ij} in $\mathbb{R}^2$ with $\det(\sigma_{ij})= a_{ij}$ such that

$$F \cap U_i \cap U_j \simeq \mathbb{C}^* \times \mathrm{Spec}\mathbb{C}[\check{\sigma}_{ij} \cap \mathbb{Z}^2],$$

moreover, $\mathrm{Spec}\mathbb{C}[\check{\sigma}_{ij} \cap \mathbb{Z}^2]$ has an $A_{a_{ij}-1}$ singularity if and only if $a_{ij} | (p_k + p_\ell)$.

(2) Let $D_i = F - U_i$. Then the followings are equivalent to each other.

(a) $D_i - D_j$ has a singularity of type A,

(b) $D_i - D_j$ has an $A_{a_j a_{ij}-1}$ singularity,

(c) $p_j | (a_{jk}p_\ell + a_{j\ell}p_k)$.

In particular, if $(\frac{p_1}{p},\frac{p_2}{p},\frac{p_3}{p},\frac{p_4}{p}) \in W_4$ and $p_j | (p - p_i)$, then

Proposition 3.4.

(1) $U_i \cap U_j \simeq \mathbb{C}^* \times \operatorname{Spec}\mathbb{C}[\check{\tau}_{ij} \cap \mathbb{Z}^3]$,

where $\tau_{ij} = \langle (0,1,0), (0,0,1), (a_{ij}, p_k, p_\ell) \rangle$.

(2) $\{z_{ij} = 0\}$ in $U_i \simeq \operatorname{Spec}\mathbb{C}[\rho_{ij} \cap \mathbb{Z}^3]$,

where $\rho_{ij} = \langle (0,1,0), (0,0,1), (p_i, p_k, p_\ell) \rangle$.

Proof. (1) Let $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4$ be generators of M such that $\mathbf{m}_i(\mathbf{e}_j) = \delta_{i,j}$. Then the cone $\check{\sigma}_i$ is expressed as

$$\check{\sigma}_i = \langle \mathbf{m}_i,\ -\frac{p_j}{a_{ij}}\mathbf{m}_i + \frac{p_i}{a_{ij}}\mathbf{m}_j,\ -\frac{p_k}{a_{ik}}\mathbf{m}_i + \frac{p_i}{a_{ik}}\mathbf{m}_k,\ -\frac{p_\ell}{a_{i\ell}}\mathbf{m}_i + \frac{p_i}{a_{i\ell}}\mathbf{m}_\ell \rangle.$$

Since g.c.d.$(p_i, p_j) = a_{ij}$, there exist integers α, β such that $-p_j \cdot \alpha + p_i \cdot \beta = a_{ij}$. Then, by the base change of M defined by

$$\mathbf{m}_i \longrightarrow \alpha \cdot \mathbf{m}_i' + \frac{p_i}{a_{ij}} \cdot \mathbf{m}_j',\quad \mathbf{m}_j \longrightarrow \beta \cdot \mathbf{m}_i' + \frac{p_j}{a_{ij}} \cdot \mathbf{m}_j',\quad \mathbf{m}_k \longrightarrow \mathbf{m}_k',\quad \mathbf{m}_\ell \longrightarrow \mathbf{m}_\ell',$$

the cone $\check{\sigma}_i$ is expressed as

$$\check{\sigma}_i = \langle \alpha \cdot \mathbf{m}_i' + \frac{p_i}{a_{ij}} \cdot \mathbf{m}_j',\ \mathbf{m}_i',\ -\frac{p_k}{a_{ik}} \cdot \alpha \cdot \mathbf{m}_i' - p_k \cdot a_{i\ell} \cdot \mathbf{m}_j' + a_{ij} \cdot a_{i\ell} \cdot \mathbf{m}_k',$$
$$-\frac{p_\ell}{a_{i\ell}} \cdot \alpha \cdot \mathbf{m}_i' - p_\ell \cdot a_{ik} \cdot \mathbf{m}_j' + a_{ij} \cdot a_{ik} \cdot \mathbf{m}_\ell' \rangle.$$

Since $U_i \cap U_j = U_i - \{z_{ij} = 0\}$ and $z_{ij} = z^{\mathbf{m}_i'}$,

$$U_i \cap U_j \simeq \mathbb{C}^* \times \operatorname{Spec}\mathbb{C}[\check{\tau}_{ij} \cap M'],$$

where M' is a free $\mathbb{Z}$-module generated by $\mathbf{m}_j', \mathbf{m}_k', \mathbf{m}_\ell'$ and

$D_i - D_j$ has an A_{p_j-1} singularity.

The next lemma is well-known.

Lemma 3.8. Let $\check{\tau}=\langle m_1, m_2\rangle$ be a 2-dimensional cone in $M_{\mathbb{R}}$. Then the singularity of $\mathrm{Spec}\,\mathbb{C}[\check{\tau}\cap M]$ is of type A if and only if $(m_1+m_2)/\det \check{\tau}$ is a element of M.

Proof of Proposition 3.7. (1) The affine torus embedding $F \cap U_i$ is written as

$$F \cap U_i = \mathrm{Spec}\,\mathbb{C}[\check{\sigma}_i' \cap M]$$

where $\check{\sigma}_i' = \langle -\frac{p_j}{a_{ij}}\cdot\mathbb{m}_i + \frac{p_i}{a_{ij}}\cdot\mathbb{m}_j\ ,\ -\frac{p_k}{a_{ik}}\cdot\mathbb{m}_i + \frac{p_i}{a_{ik}}\cdot\mathbb{m}_k\ ,\ -\frac{p_\ell}{a_{i\ell}}\cdot\mathbb{m}_i + \frac{p_i}{a_{i\ell}}\cdot\mathbb{m}_\ell\rangle$.

By the base change of M defined in the proof of Proposotion 3.4.,

$$\check{\sigma}_i' = \langle\ \mathbb{m}_i\ ,\ -\frac{p_k}{a_{ik}}\cdot\alpha\cdot\mathbb{m}_i - p_k a_{i\ell}\cdot\mathbb{m}_j' + a_{ij}a_{i\ell}\cdot\mathbb{m}_k'\ ,$$

$$-\frac{p_\ell}{a_{i\ell}}\cdot\alpha\cdot\mathbb{m}_i' - p_\ell a_{ik}\cdot\mathbb{m}_j' + a_{ij}a_{ik}\cdot\mathbb{m}_\ell'\ \rangle.$$

Since $F \cap U_i \cap U_j = (F \cap U_i) - \{z_{ij}=0\}$, and $z_{ij}=z^{\mathbb{m}_i'}$, we have

$$F \cap U_i \cap U_j = \mathbb{C}^* \times \mathrm{Spec}\,\mathbb{C}[\ \check{\sigma}_{ij}' \cap M\ ]$$

where $\check{\sigma}_{ij}' = \langle -p_k\cdot\mathbb{m}_j' + a_{ij}\cdot\mathbb{m}_k'\ ,\ -p_\ell\cdot\mathbb{m}_j' + a_{ij}\cdot\mathbb{m}_\ell\ \rangle$.

Thus the assertion follows from Lemma 3.8.

(2) Similarly, $D_i - D_j$ is written as

$$D_i - D_j = \operatorname{Spec}\mathbb{C}[\ \check{\tau}'_{ij} \cap M\ ]$$

where $\check{\tau}'_{ij} = \left\langle -\dfrac{p_k}{a_{ik}}\cdot\mathbf{m}_i + \dfrac{p_i}{a_{ik}}\cdot\mathbf{m}_k\ ,\ -\dfrac{p_\ell}{a_{i\ell}}\cdot\mathbf{m}_i + \dfrac{p_i}{a_{i\ell}}\cdot\mathbf{m}_\ell \right\rangle.$

Since g.c.d.$(p_j, p_k, p_\ell) = 1$, we have $\det \check{\tau}'_{ij} = a_j a_{ij}$. This shows the equivalence between (a) and (b). By Lemma 3.8., the conditions (a) and (c) are also equivalent to each other.

Q.E.D.

§4. Singularities on E

Let $f \in \mathbb{C}[z_1, z_2, z_3, z_4]$ be a non-degenerate polynomial which defines a simple K3 singularity at the origin $0 \in \mathbb{C}^4$. We set $X = \{f = 0\} \subset \mathbb{C}^4$, $x = 0 \in \mathbb{C}^4$ and let $\pi : (\tilde{X}, E) \longrightarrow (X, x)$ be the minimal resolution of (X, x) constructed in §3. In this section, we investigate the singularities on the normal K3 surface E.

In the following, we assume that f satisfy the condition (*) below.

> (*) For any i, f_0 contains a term of the form $z_i{}^n$ or $z_i{}^n z_j$ with a non-zero coefficient. (We say f contains z^ν if $f = \sum_\nu a_\nu z^\nu$ and $a_\nu \neq 0$.)

By Lemma 2.3.(1)., there exists f which satisfies (*) for any weight α of W_4. If f is semi-quasihomogeneous, then the condition (*) is satisfied, in particular, the polynomials in Table 2.2. satisfy (*).

Lemma 4.1.

(1) E has γ_{ij} singular points of type $A_{a_{ij}-1}$.

where $\gamma_{ij} = \#\{\nu \in \Delta_0 \cap \mathbb{Z}^4 \mid \nu_k = \nu_\ell = 0\} - 1$.

(2) If f_0 contains $z_i{}^m z_j$ and does not $z_i{}^n$, then E has a singular point of type $A_{p_i - 1}$.

(3) Any singular point on E belongs to (1) or (2).

Proof. Here we use the notation introduced in §3. Let $L_{ij} = D_k \cap D_\ell$ $(=\{z_k = z_\ell = 0\}$ in $F)$. Then L_{ij} is a projective line $\mathbb{P}^1(\mathbb{C})$ on F and any singular point on E is in L_{ij} for some i,j. We divide L_{ij} into two points $P_i = L_{ij} \cap D_j$, $P_j = L_{ij} \cap D_i$ and one dimensional complex torus $T_{ij} = L_{ij} - \{P_i, P_j\} \simeq \mathbb{C}^*$. Since f is non-degenerate, there is an embedded resolution $\rho:(Y,M) \longrightarrow (X,x)$ such that Y is a torus embedding associated with a simplicial subdivision Σ^* of $\Gamma^*(f)$, i.e., Σ^* is a fan of cones generated by a part of basis of N and for any cone $\sigma \in \Sigma^*$ there exists a cone $\tau \in \Gamma^*(f)$ with $\sigma \subset \tau$. The resolution ρ dominates the minimal resolution π, so there exists a resolution $\varphi:(Y,M) \longrightarrow (\tilde{X},E)$ with $\rho = \pi \circ \varphi$.

$$\begin{tikzcd}
(Y,M) \arrow[rr, "\varphi"] \arrow[dr, "\rho"'] & & (\tilde{X},E) \arrow[dl, "\pi"] \\
& (X,x) &
\end{tikzcd}$$

Choose a sufficiently small neighbourhood E_{ij} of T_{ij} in E, and let $\varphi_{ij}:(\varphi^{-1}(E_{ij}), C_{ij}) \longrightarrow (E_{ij}, \mathrm{Sing}(E_{ij}))$ be the resolution of E_{ij} obtained by restricting φ to $\varphi^{-1}(E_{ij})$. Then, by M. Oka ([9],Lemma 4.8. and its proof.), φ_{ij} is locally γ_{ij} copies of the resolution of $\mathrm{Spec}\,\mathbb{C}[\check{\sigma}_{ij} \cap \mathbb{Z}^3]$, where σ_{ij} is a cone in Proposition 3.7.(1). By Lemma 2.3.(3), $a_{ij} | (p_k + p_\ell)$ and hence assertion (1) is proved. If f_0 contains $z_i^m z_j$ and does not z_i^n, then $P_i \in E$. Let E_i be a sufficiently small neighbourhood of P_i in E, and $\varphi_i : (\varphi^{-1}(E_i), C_i) \longrightarrow (E_i, P_i)$ be the resolution as above. Then φ_i is locally isomorphic to the resolution of $D_j - D_i$. Thus the assertion (2) follows from Proposition 3.7.(2).

Q.E.D.

Theorem 4.2. Let f be a non-degenerate polynomial which defines a simple K3 singularity, and assume that f satisfies the condition (*). Let $\pi : (\tilde{X}, E) \longrightarrow (X, x)$ be the minimal resolution. Then the type and the number of singularities on E are determined by the weight $\alpha(f)$ and not depend on the choice of f. In particular, E has t_{ij} singular points of type $A_{a_{ij}-1}$ and σ_i singular points of type A_{p_i-1}, where

$$t_{ij} = \#\{\nu \in T(\alpha) \mid \nu_k = \nu_\ell = 0\} - 1, \text{ and}$$

$$\sigma_i = \begin{cases} 0 & \text{if } p \mid p_i \\ 1 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 4.1., The singularities on E are determined by the Newton boundary of f_0. Let α be a weight in W_4 and let f be a polynomial with $\alpha(f) = \alpha$ and $\Gamma(f_0)$ = the convex hull of $T(\alpha)$. Then the singularities on E are coinside with those stated in this theorem. Thus we may show that if g is a polynomial with $\alpha(g) = \alpha$, E' is the exceptional set of the resolution of $\{g = 0\}$, then the singularities on E and on E' are same. By the condition (*), one of the following cases occurs for each $i=1,2,3,4$. Let $\{i,1,k,\ell\}$ be the set of indices $\{1,2,3,4\}$.

Case(1) $p_i | (p-p_j)$, and $p_i | (p-p_k)$.

Case(2) $p_i | (p-p_j)$, $p_i \nmid (p-p_k)$, and $p_i \nmid (p-p_\ell)$.

Case(3) $p_i | p$, $p_i \nmid (p-p_j)$, $p_i \nmid (p-p_k)$, and $p_i \nmid (p-p_\ell)$.

For each case, we study singularities on $E \cap V_i$ and on $E' \cap V_i$ using Lemma 3.2. and Propositon 3.6. Recall that the singularities on $E \cap V_i$ are contained in $\{P_i\} \cup T_{ij} \cup T_{ik} \cup T_{i\ell}$, here we use the same notation in the proof of lemma 4.1.

Case(1) In this case, Both $E \cap V_i$ and $E' \cap V_i$ are non-singular.

Case(2) In this case, E (resp. E') has no singular points on $V_i - T_{ij}$ (resp. $V_i - L_{ij}$), and the type of singularities on T_{ij} and of a singularity P_i are same. Now we consider the singularities on $E \cap V_j$ and on $E' \cap V_j$. By the condition (*) again, we have the following 3 cases.

Case(2-1) Assume $p_j | (p-p_k)$ or $p_j | (p-p_\ell)$. Then E and E' have no singular points on T_{ij} , and hence $E \cap V_i$ and $E' \cap V_i$ are non-singular.

Case(2-2) Assume $p_j | (p-p_i)$, $p_j \nmid (p-p_k)$, and $p_j \nmid (p-p_\ell)$. Then E and E' have no singular points on T_{jk} and on $T_{j\ell}$.

Since the type of the singularities on $\{P_i\} \cup T_{ij} \cup \{P_j\}$ are same, both $E \cap (V_i \cup V_j)$ and $E' \cap (V_i \cup V_j)$ have t_{ij} singular points of type $A_{a_{ij}-1}$.

Case(2-3) Assume $p_j \nmid (p-p_i)$, $p_j \nmid (p-p_k)$, and $p_j \nmid (p-p_\ell)$. Then the point P_j is not contained in E and E' , and thus both $E \cap V_i$ and $E' \cap V_i$ have t_{ij} singular points of type $A_{a_{ij}-1}$.

Case(3) In this case, the point P_i is not on E and E' . So we have $\mathrm{Sing}(E) = \mathrm{Sing}(E \cap (V_i \cup V_k \cup V_\ell))$ and $\mathrm{Sing}(E') = \mathrm{Sing}(E \cap (V_j \cup V_k \cup V_\ell))$. If Case(3) occurs for i and j , then both E and E' have t_{ij} singular points of type $A_{a_{ij}-1}$ on T_{ij} .

By the above discussion, we obtain the assertion.

Q.E.D.

Remark 4.3. The condition (*) is essential for Theorem 4.2. For example, consider the polynomials:

$$f = x^4 + y^4 + z^4 + (x^2 + y^2 + z^2)w^2 + w^5$$

$$g = x^4 + y^4 + z^4 + (x^2w + y^2w + z^3)w + w^5.$$

Then we have $\alpha(f) = \alpha(g) = \alpha = (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$ but the K3 surface E for f (resp. g) has A_1-singularity (resp. A_2-singularity) at the point P_4 while E for a polynomial h which satisfies (*) and $\alpha(h) = \alpha$ is non-singular.

By Theorem 4.2., the singularities on the normal K3 surface E are determined by the weight $\alpha=\alpha(f)$. So we denote by $r(\alpha)$ the rank of $\mathrm{Sing}(E)$, i.e.,

$$r(\alpha) = \sum_{i<j} t_{ij}\cdot(a_{ij}-1) + \sum_{i=1}^{4} \sigma_i(p_i-1) .$$

Let $t(\alpha) = \#T(\alpha)$. Then the polynomial f with $\alpha(f) = \alpha$ has $t(\alpha)$ terms in general, in other words, f is of the form

$$f = \sum_{\nu\in T(\alpha)} a_\nu z^\nu \quad , \quad a_\nu \neq 0 \quad \text{for all } \nu.$$

For general f, there is a coordinate change which preserves the weight α,

$$z_i \longrightarrow \sum_{\nu\in N_i(\alpha)} b_{i\nu} w^\nu \quad , \quad N_i(\alpha) = \{\nu\in T(\alpha) \mid \sum_{j=1}^{4} \nu_j p_j = p_i\}$$

such that $f(w)$ has $t(\alpha)-n(\alpha)+4$ terms, of which 4 terms have coefficients 1, where

$$n(\alpha) = \sum_{i=1}^{4} \#N_i(\alpha).$$

Then we obtain the following relation.

Corollary 4.4. For any weight $\alpha \in W_4$, we have

$$t(\alpha) - n(\alpha) + r(\alpha) = 19.$$

Remark 4.5. We may consider that the number $t(\alpha)-n(\alpha)$ is the number of parameters in a polynomial f with $\alpha(f) = \alpha$. Then Corollary 4.4. suggests us that $t(\alpha)-n(\alpha)$ parameters associate to the moduli of K3 surface E with fixed singularities of rank $r(\alpha)$.

Table 4.6.

No.	$(p_1,p_2,p_3,p_4:p)$	$t(\alpha)$	$n(\alpha)$	Sing(E)	$r(\alpha)$
1	(1,1,1,1:4)	35	16	non-singular	0
2	(4,3,3,2:12)	15	7	$3A_1+4A_2$	11
3	(2,2,1,1:6)	30	14	$3A_1$	3
4	(4,4,3,1:12)	21	11	$3A_3$	9
5	(3,1,1,1:6)	39	20	non-singular	0
6	(5,2,2,1:10)	28	14	$5A_1$	5
7	(4,2,1,1:8)	35	18	$2A_1$	2
8	(6,3,2,1:12)	27	14	$2A_1+2A_2$	6
9	(10,5,4,1:20)	23	13	A_1+2A_4	9
10	(6,4,1,1:12)	39	21	A_1	1
11	(15,10,3,2:30)	18	10	$3A_1+2A_2+A_4$	11
12	(9,6,2,1:18)	30	16	$3A_1+A_2$	5
13	(12,8,3,1:24)	27	15	$2A_2+A_3$	7
14	(21,14,6,1:42)	24	14	$A_1+A_2+A_6$	9
15	(5,4,3,3:15)	12	6	$5A_2+A_3$	13
16	(8,7,6,3:24)	9	5	$A_1+4A_2+A_6$	15
17	(5,5,3,2:15)	14	8	A_1+3A_4	13
18	(3,3,2,1:9)	23	11	A_1+3A_2	7
19	(3,2,2,1:8)	24	11	$4A_1+A_2$	6
20	(9,8,6,1:24)	18	10	$A_1+A_2+A_8$	11
21	(2,1,1,1:5)	34	16	A_1	1
22	(6,5,3,1:15)	21	11	$2A_2+A_5$	9
23	(5,3,2,2:12)	17	8	$6A_1+A_4$	10
24	(5,4,2,1:12)	24	12	$3A_1+A_4$	7

25	(4,3,1,1:9)	33	17	A_3	3
26	(9,5,4,2:20)	13	7	$5A_1+A_8$	13
27	(11,8,3,2:24)	15	9	$3A_1+A_{10}$	13
28	(10,7,3,1:21)	24	14	A_9	9
29	(15,6,5,4:30)	10	6	$2A_1+A_2+A_3+2A_4$	15
30	(20,8,7,5:40)	8	6	$A_3+2A_4+A_6$	17
31	(12,5,4,3:24)	12	7	$2A_2+2A_3+A_4$	14
32	(7,3,2,2:14)	19	9	$7A_1+A_2$	9
33	(9,4,3,2:18)	16	8	$4A_1+2A_2+A_3$	11
34	(15,7,6,2:30)	13	7	$5A_1+A_2+A_6$	13
35	(14,7,4,3:28)	12	8	$A_1+A_2+2A_6$	15
36	(10,5,3,2:20)	16	9	$2A_1+A_2+2A_4$	12
37	(8,4,3,1:16)	24	13	A_2+2A_3	8
38	(15,8,6,1:30)	21	12	$A_1+A_2+A_7$	10
39	(9,5,3,1:18)	24	13	$2A_2+A_4$	8
40	(7,4,2,1:14)	27	14	$3A_1+A_3$	6
41	(12,7,3,2:24)	16	9	$2A_1+2A_2+A_6$	12
42	(5,3,1,1:10)	36	19	A_2	2
43	(18,11,4,3:36)	12	8	$A_1+2A_2+A_{10}$	15
44	(8,5,2,1:16)	28	15	$2A_1+A_4$	6
45	(14,9,4,1:28)	24	14	A_1+A_8	9
46	(33,22,6,5:66)	9	7	$A_1+A_2+A_4+A_{10}$	17
47	(21,14,4,3:42)	13	8	$A_1+2A_2+A_3+A_6$	14
48	(24,16,5,3:48)	12	8	$2A_2+A_4+A_7$	15
49	(21,14,5,2:42)	15	9	$3A_1+A_4+A_6$	13
50	(15,10,4,1:30)	25	14	$A_1+A_3+A_4$	8
51	(18,12,5,1:36)	24	14	A_4+A_5	9

52	(12,9,8,7:36)	5	4	$A_2+A_3+A_6+A_7$	18
53	(6,5,4,3:18)	10	5	$A_1+3A_2+A_3+A_4$	14
54	(7,6,5,3:21)	9	5	$3A_2+A_4+A_5$	15
55	(7,6,5,2:20)	11	6	$3A_1+A_5+A_6$	14
56	(11,8,6,5:30)	6	5	$A_1+A_7+A_{10}$	18
57	(9,6,5,4:24)	8	5	$2A_1+A_2+A_4+A_8$	16
58	(6,5,4,1:16)	19	10	$A_1+A_4+A_5$	10
59	(8,7,5,1:21)	18	10	A_4+A_7	11
60	(7,6,4,1:18)	19	10	$A_1+A_3+A_6$	10
61	(11,7,6,4:28)	7	5	$2A_1+A_5+A_{10}$	17
62	(8,5,4,3:20)	10	6	$A_2+2A_3+A_7$	15
63	(4,3,2,1:10)	23	11	$2A_1+A_2+A_3$	7
64	(10,7,4,3:24)	10	7	$A_1+A_6+A_9$	16
65	(14,11,5,3:33)	9	7	A_4+A_{13}	17
66	(3,2,1,1:7)	31	15	A_1+A_2	3
67	(9,7,3,2:21)	14	8	$A_1+2A_2+A_8$	13
68	(13,10,4,3:30)	10	7	$A_1+A_3+A_{12}$	16
69	(7,4,3,2:16)	14	7	$4A_1+A_2+A_6$	12
70	(8,5,3,2:18)	14	8	$2A_1+A_4+A_7$	13
71	(7,4,3,1:15)	22	12	A_3+A_6	9
72	(7,5,2,1:15)	26	14	A_1+A_6	7
73	(25,10,8,7:50)	6	5	$A_1+A_4+A_6+A_7$	18
74	(16,7,5,4:32)	9	6	$2A_3+A_4+A_6$	16
75	(11,5,4,2:22)	14	7	$5A_1+A_3+A_4$	12
76	(13,6,5,2:26)	13	7	$4A_1+A_4+A_5$	13
77	(13,7,5,1:26)	21	12	A_4+A_6	10
78	(11,6,4,1:22)	22	12	$A_1+A_3+A_5$	9

79	(16,9,5,2:32)	13	8	$2A_1+A_4+A_8$	14
80	(22,13,5,4:44)	9	7	$A_1+A_4+A_{12}$	17
81	(13,8,3,2:26)	16	9	$3A_1+A_2+A_7$	12
82	(11,7,3,1:22)	25	14	A_2+A_6	8
83	(27,18,5,4:54)	10	7	$A_1+A_3+A_4+A_8$	16
84	(9,7,6,5:27)	6	4	$A_2+A_4+A_5+A_6$	17
85	(5,4,3,2:14)	13	6	$3A_1+A_2+A_3+A_4$	12
86	(9,7,5,4:25)	7	5	$A_3+A_6+A_8$	17
87	(5,4,3,1:13)	20	10	$A_2+A_3+A_4$	9
88	(11,9,5,2:27)	11	7	$A_1+A_4+A_{10}$	15
89	(5,3,2,1:11)	24	12	$A_1+A_2+A_4$	7
90	(17,7,6,4:34)	8	5	$2A_1+A_3+A_5+A_6$	16
91	(19,8,6,5:38)	7	5	$A_1+A_4+A_5+A_7$	17
92	(19,11,5,3:38)	10	7	$A_2+A_4+A_{10}$	16
93	(17,10,4,3:34)	11	7	$A_1+A_2+A_3+A_9$	15
94	(7,5,4,3:19)	9	5	$A_2+A_3+A_4+A_6$	15
95	(7,5,3,2:17)	13	7	$A_1+A_2+A_4+A_6$	13

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Riemann-Roch theorem for normal isolated singularities

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Dedicated to Professor Yukihiro Kodama on his 60th birthday

§0. Introduction

In this paper we show that the analogous formula of Riemann-Roch also holds for normal isolated singularities of arbitrary dimensions. This is an answer to the paper of Kato [Ka]. Kato proved that the theorem of Riemann-Roch type holds for strongly pseudoconvex manifolds of dimension 2. It has been expected that there exists the formula which holds for strongly pseudoconvex manifolds of arbitrary dimensions.

On the latter half of this article three typical applications of generalized Kato's Riemann-Roch theorem are summarized. The first application of the theorem is to study the geometric genus of normal isolated singularities of odd dimensions. For example, the threefold case, the following theorem folds:

Theorem. Let $\pi : Y \longrightarrow X$ be a good resolution of normal three dimensional isolated singularity (X,x). Denote by A_i $(i = 1,\dots,r)$ the irreducible component of the exceptional set $E = \pi^{-1}(\{x\})$. Suppose that (X,x) is Gorenstein. Then the canonical divisor K_Y is given by

$$K_Y \sim \sum_{i=1}^{r} \lambda_i A_i$$

for some integers λ_i's. Let $p_g(X,x)$ be the geometric genus of (X,x). Then

$$24p_g(X,x) = \sum_{i=1}^{r} \lambda_i\{c_2(A_i) + c_1(A_i)c_1(N_{A_i})\},$$

where N_{A_i} is the normal bundle of A_i in Y.

The second uses of the theorem is to calculate L^2-plurigenera $\{ \delta_m \}$ of normal isolated singularities. δ_1 is the geometric genus of a singularity. The third uses of the theorem is to calculate other plurigenera $\{ d_m \}$ of normal isolated singularities, which was introduced by Ishii [I]. d_1 is the geometric genus of the exceptional set of a resolution.

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§1. Preliminaries

Let (X,x) be a germ of an n-dimensional normal isolated singularity. By a theorem of Artin [A], (X, x) can be realized as a Zariski open subset of a projective variety Y with x as its only singularity. Let $\pi : M \longrightarrow X$ be a good resolution of the singular point. Then, in a natural manner, we get a desingularization $\rho : N \longrightarrow Y$ of Y by letting N to be $(Y - \{x\}) \cup M$. Let $E = \pi^{-1}(\{x\})$ and denote by D_i $(i=1,\ldots,r)$ the irreducible componets of E.

These notations are used throghout the paper.

Note that M is a strongly pseudoconvex manifold and N is a non-singular compactification of M. We may also assume that $N - M$ consists of non-singular divisors in normal crossings.

Let D be a non-singular divisor of $M \subset N$, and let $d \in H^2(N,\mathbb{Z})$ be the cohomology class represented by the oriented $(2n-2)$-cycle D. Denote by $[D]$ the line bundle defined by the integral divisor D. Then $c_1([D]) = d$.

The natural orientation of N defines an element of the $2n$-dimensional integral homology group $H_{2n}(N,\mathbb{Z})$ called the fundamental cycle of N.

In general, following the notation in [Hi], for $a = \sum_{k=0}^{n} a_k \in H^*(N,\mathbb{C})$ with $a_k \in H^{2k}(N,\mathbb{C})$, we put

$$\kappa_n[a] = a_{2n},$$
$$\kappa_n(a) = a_{2n}[N] = \langle a_{2n}, [N] \rangle,$$

$[N]$ denoting the fundamental $2n$-cycle of N.

Let $j : D \longrightarrow N$ be the embedding of D in N, and $c_i \in H^{2i}(N,\mathbb{Z})$ be the Chern classes of N. Every product $c_{j_1} c_{j_2} \cdots c_{j_r}$ of weight $n-s = j_1 + j_2 + \cdots + j_r$ defines an integer $c_{j_1} c_{j_2} \cdots c_{j_r} d^s[N]$, which is equal to $\langle j^*(c_{j_1} c_{j_2} \cdots c_{j_r} d^{s-1}), [D] \rangle$ if $s \geq 1$.

Denote the complex analytic tangent bundles of N, D by T_N, T_D. There is an exact sequence

$$0 \longrightarrow T_D \longrightarrow T_N|_D \longrightarrow [D]|_D \longrightarrow 0,$$

so we have

$$j^*c(T_N) = c(j^*T_N) = c(T_N|_D) = c(T_D)(1 + j^*d).$$

(multiplicity of the total Chern class)

Then any $j^*c_j(N)$ can be represented by the Chern classes of D and j^*d. Thus $(c_{j_1}c_{j_2}\cdots c_{j_r}d^s)[N]$ is independent of the choice of affine models of (X,x) and their non-singular compactifications if $s \geq 1$.

Let f be a two dimensional cohomology class of $H^2(N,\mathbb{Z})$. Define $T(N,f)$ by

$$T(N,f) = < \kappa_n\left[e^f \prod_{i=1}^{n} \frac{\gamma_i}{1-e^{-\gamma_i}}\right], [N] >.$$

This formula is to be understood as follows: There is a formal factorization

$$1+c_1x + \cdots + c_nx^n = (1+\gamma_1x)\cdots(1+\gamma_nx).$$

, where $c_i \in H^{2i}(N,\mathbb{Z})$ are the Chern classes of N. Consider the term of degree n in f and the γ_i of the expression in square brackets. It is a symmetric function in the γ_i and is therefore a polynomial in f and the c_i with rational coefficients. If the multiplication is interpreted as the cup product in $H^*(N,\mathbb{Z})$, this polynomial defines as an element of $H^{2n}(N,\mathbb{Z})\otimes\mathbb{Q}$. The value of this element on the 2n-dimensional cycle of N determined by the natural orientation is denoted by $T(N,f)$.

The relative cohomology $H^n(N,M;\mathbb{R}) = H^n(N,M)$ carries a mixed Hodge structure such that the exact sequence

$$\cdots \longrightarrow H^n(N) \longrightarrow H^n(M) \longrightarrow H^{n+1}(N,M) \longrightarrow H^{n+1}(N) \longrightarrow \cdots$$

is an exact sequence of the mixed Hodge structures. The mixed Hodge structure $H^n(N,M)$ is called the Deligne mixed Hodge structure. We denote by $H_n^{p,q}(N,M)$ the spaces of

$Gr^W_{p+q}(H^n(N,M))$ and let $h_n^{p,q}(X) = \dim_{\mathbb{C}}(H_n^{p,q}(X))$ (the Hodge numbers).

Lemma 1.1. Let the notation be as above. Then

$$H_2^{1,1}(N) \longrightarrow H_2^{1,1}(M)$$

is surjective.

Proof. $M - N$ consists of finite number of irreducible non-singular divisors B_i $(i = 1, \cdots, s)$, $N - M = \bigcup_{i=1}^{s} B_i$. Consider the following exact sequence of cohomologies with coefficient $\mathbb{Z}$:

$$H^2(N-B_1\cup\cdots\cup B_{k-1}) \longrightarrow H^2(N-B_1\cup\cdots\cup B_{k-1}\cup B_k)$$
$$\longrightarrow H^3(N-B_1\cup\cdots\cup B_{k-1}, N-B_1\cup\cdots\cup B_{k-1}\cup B_k),$$

where $1 \leq k \leq s$ and in the case $k = 1$, $N-B_1\cup\cdots\cup B_{k-1} = N$. Then one obtains an exact sequence:

$$H^2(N-B_1\cup\cdots\cup B_{k-1}) \longrightarrow H^2(N-B_1\cup\cdots\cup B_{k-1}\cup B_k)$$
$$\longrightarrow H^1(B_k-B_1\cup\cdots\cup B_{k-1})(-1)$$

as in [D].

Since $B_k-B_1\cup\cdots\cup B_{k-1}$ is non-singular,

$$H_1^{0,0}(B_k-B_1\cup\cdots\cup B_{k-1}) = 0.$$

Hence the morphism

$$H_2^{1,1}(N-B_1\cup\cdots\cup B_{k-1}) \longrightarrow H_2^{1,1}(N-B_1\cup\cdots\cup B_{k-1}\cup B_k)$$

is surjective. Thus

$$H_2^{1,1}(N) \longrightarrow H_2^{1,1}(M)$$

is also surjective.

Lemma 1.2. Let F be a line bundle on M. Then there exists an element $\overline{f} \in H^2(N,\mathbb{C})$ such that $j_k{}^*(\overline{f}) = c_1(F|_{D_k})_{\mathbb{C}}$ for any k, where $c_1(F|_{D_k})_{\mathbb{C}}$ is the image of the canonical map

$$H^2(D_k,\mathbb{Z}) \longrightarrow H^2(D_k,\mathbb{C}).$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
 & H^1(M,\mathcal{O}^*) \arrow[r] \arrow[d, "\delta_1"] & H^1(D_k,\mathcal{O}^*) \arrow[d] \\
H^2(N,\mathbb{Z}) \arrow[r] \arrow[d, "\nu"] & H^2(M,\mathbb{Z}) \arrow[r] \arrow[d, "\mu"] & H^2(D_k,\mathbb{Z}) \arrow[d, "\lambda_k"] \\
H^2(N,\mathbb{C}) \arrow[r, "j^*"'] & H^2(M,\mathbb{C}) \arrow[r, "i_k^*"'] & H^2(D_k,\mathbb{C}) .
\end{tikzcd}$$

Then $\lambda_k(c_1(F|_{D_k})) = i_k^*(\mu(\delta_1(F)))$. As is well-known, $\lambda_k(c_1(F|_{D_k}))$ is of type $(1,1)$. The canonical morphism i_k^* is a morphism of mixed Hodge structures, so it preserves the type of cohomology. Let $f \in H^2(M,\mathbb{C})$ be a $(1,1)$-component of $\mu(\delta_1((F))$. Then $i_k^*(f) = i_k^*(\mu(\delta_1(F)))$. From the above lemma there is an element $\overline{f} \in H_2^{1,1}(N)$ such that $j^*(\overline{f}) = f$. Hence $j_k^*(\overline{f}) = i_k^* \circ j^*(\overline{f}) = \lambda_k(c_1(F|_{D_k}))$.

Let $E = \sum D_k$ be the decomposition of the exceptional set E into its irreducible components. Let $j_k : D_k \longrightarrow N$ be the natural map. Because of the exact sequence

$$0 \longrightarrow T_{D_k} \longrightarrow T_V|_{D_k} \longrightarrow [D_k]|_{D_k} \longrightarrow 0$$

we have

$$j_k^* c(V) = c(D_k)\cdot(1 + j_k^* d_k)$$

in $H^*(N,\mathbb{Z})$, where $d_k = c_1([D_k]) \in H^2(N,\mathbb{Z})$. Let

$$1 + c_1(N)x + \cdots + c_n(N)x^n = (1+\gamma_1 x)(1+\gamma_2 x)\cdots(1+\gamma_n x)$$

be a formal factrization of the total Chern classes of N.

Then

$$\begin{aligned}
&j_k^*\{(1+\gamma_1 x)\cdots(1+\gamma_n x)\}\\
&= j_k^*\{1 + c_1(N)x + \cdots + c_n(N)x^n\}\\
&= \{1 + c_1(D_k)x + \cdots + c_{n-1}(D_k)x^{n-1}\}(1 + j_k^* d_k x)\\
&= (1+\delta_1^{(k)}x)\cdots(1+\delta_{n-1}^{(k)}x)(1 + j_k^* d_k x)
\end{aligned}$$

gives a formal factorization of $j_k^* c(N)$.

Lemma 1.3. Let $D = \sum \lambda_k D_k$ ($\lambda_k \in \mathbb{Z}$) be an integral divisor with the first Chern class d on N. Then

$$< \kappa_n\left[e^{\overline{f}} \prod_{i=1}^{n} \frac{\gamma_i}{1-e^{-\gamma_i}}\right] - \kappa_n\left[e^{\overline{f}+d} \prod_{i=1}^{n} \frac{\gamma_i}{1-e^{-\gamma_i}}\right], [N] >$$

is independent of the choice of $\overline{f}$.

Proof.

$$\begin{aligned}
&< \kappa_n\left[e^{\overline{f}} \prod_{i=1}^{n} \frac{\gamma_i}{1-e^{-\gamma_i}}\right] - \kappa_n\left[e^{\overline{f}+d} \prod_{i=1}^{n} \frac{\gamma_i}{1-e^{-\gamma_i}}\right], [N] >\\
&= < - \kappa_{n-1}\left[e^{\overline{f}+d}\left(\frac{1-e^{-d}}{d}\right) \prod_{i=1}^{n} \frac{\gamma_i}{1-e^{-\gamma_i}}\right] d , [N] >\\
&= - < \kappa_{n-1}\left[e^{\overline{f}+d}\left(\frac{1-e^{-d}}{d}\right) \prod_{i=1}^{n} \frac{\gamma_i}{1-e^{-\gamma_i}}\right], [D] >\\
&= - \sum_k \lambda_k < \kappa_{n-1}\left[e^{\overline{f}+d}\left(\frac{1-e^{-d}}{d}\right) \prod_{i=1}^{n} \frac{\gamma_i}{1-e^{-\gamma_i}}\right], [D_k] >\\
&= - \sum_k \lambda_k < j_k^*\{\kappa_{n-1}\left[e^{\overline{f}+d}\left(\frac{1-e^{-d}}{d}\right) \prod_{i=1}^{n} \frac{\gamma_i}{1-e^{-\gamma_i}}\right]\}, [D_k] >
\end{aligned}$$

$$= - \sum_k \lambda_k < \kappa_{n-1}\left[e^{j_k^*(\bar{f}+d)}\left(\frac{1 - e^{-j_k^* d}}{j_k^* d}\right)\prod_{i=1}^{n-1}\frac{\delta_i^{(k)}}{1 - e^{-\delta_i^{(k)}}}\cdot\frac{j_k^* d_k}{1 - e^{-j_k^* d_k}}\right], [D_k]>$$

Following Laufer [L], we consider the sheaf cohomology with support at infinity. Let F be a line bundle on M. The sequence

$$0 \longrightarrow \Gamma(M,\mathcal{O}(F)) \longrightarrow \Gamma_\infty(M,\mathcal{O}(F)) \longrightarrow H_*^1(M,\mathcal{O}(F)) \longrightarrow \cdots$$

is exact. By Siu [Si], p. 374, any section of F defined near the boundary of M has an analytic continuation to $M - E$. Therefore there is a natural isomorphism $\Gamma_\infty(M,\mathcal{O}(F)) \simeq \Gamma(M-E,\mathcal{O}(F))$. By Hartshorne [H], p. 225, there exists an isomorphism:

$$H_*^1(M,\mathcal{O}(F)) \simeq H^1(M,\mathcal{O}(K-F))$$

where K denote the line bundle determined by canonical divisors. Since M is strongly pseudoconvex, $H^1(M,\mathcal{O}(K-F))$ is finite dimendional. Hence by the inequality

$$\begin{aligned}\dim \Gamma(M-E,\mathcal{O}(F))/\Gamma(M,\mathcal{O}(F)) &\leq \dim H_*^1(M,\mathcal{O}(F))\\ &= \dim H^1(M,\mathcal{O}(K-F),\end{aligned}$$

we have $\dim \Gamma(M-E,\mathcal{O}(F))/\Gamma(M,\mathcal{O}(F)) \leq +\infty$. We define the Euler-Poincaré characteristic $\chi(M,\mathcal{O}(F))$ by

$$\chi(M,\mathcal{O}(F)) = \dim \Gamma(M-E,\mathcal{O}(F))/\Gamma(M,\mathcal{O}(F)) - \sum_{q=1}^{\infty}(-1)^q \dim H^q(M,\mathcal{O}(F)).$$

Now we shall prove the following.

Theorem 1.4 (Riemann-Roch theorem for integral divisors). Let F be any line bundle on M. Then, for any integral divisor D with the first Chern class d on M, the equality

$$\chi(M,\mathcal{O}(F+[D])) - \chi(M,\mathcal{O}(F)) = T(N,\bar{f}) - T(N,\bar{f}+d) \qquad (*)$$

holds, where $\bar{f}$ is an extension of the (1,1)-Hodge component of $\delta(F)_{\mathbb{C}} \in H^2(M,\mathbb{C})$.

Proof. We borrow the argument of Kato [Ka], p.245. Let A be any non-singular divisor in M, i.e., A is an irreducible component of the exceptional set E.

We consider the exact sequence

$$0 \longrightarrow \mathcal{O}(F-[A]) \longrightarrow \mathcal{O}(F) \longrightarrow \mathcal{O}_A(F) \longrightarrow 0 \qquad \text{on } M.$$

Then we get the exact sequence

$$\begin{aligned} 0 &\longrightarrow \Gamma(M,\mathcal{O}(F-[A])) \xrightarrow{\varphi_1} \Gamma(M,\mathcal{O}(F)) \longrightarrow \Gamma(A,\mathcal{O}_A(F)) \\ &\longrightarrow H^1(M,\mathcal{O}(F-[A])) \longrightarrow H^1(M,\mathcal{O}(F)) \longrightarrow H^1(A,\mathcal{O}_A(F)) \\ &\longrightarrow \cdots \longrightarrow \\ &\longrightarrow H^{n-1}(M,\mathcal{O}(F-[A])) \longrightarrow H^{n-1}(M,\mathcal{O}(F)) \longrightarrow H^{n-1}(A,\mathcal{O}_A(F)) \\ &\longrightarrow H^n(M,\mathcal{O}(F-[A])) \longrightarrow H^n(M,\mathcal{O}(F)) \longrightarrow 0 . \end{aligned}$$

Hence $\dim \Gamma(M,\mathcal{O}(F))/\varphi_1\Gamma(M,\mathcal{O}(F-[A])) \leq \dim \Gamma(A,\mathcal{O}_A(F)) < \infty$.

Moreover

$$\begin{aligned} &\dim \Gamma(M,\mathcal{O}(F))/\varphi_1\Gamma(M,\mathcal{O}(F-[A])) - \dim \Gamma(A,\mathcal{O}_A(F)) + \\ &+ \dim H^1(M,\mathcal{O}(F-[A])) - \dim H^1(M,\mathcal{O}(F)) + \dim H^1(A,\mathcal{O}_A(F)) \\ &- \cdots + \cdots - \cdots \\ &+ (-1)^n\dim H^{n-1}(M,\mathcal{O}(F-[A])) - (-1)^n\dim H^{n-1}(M,\mathcal{O}(F)) \\ &\qquad\qquad + (-1)^n\dim H^{n-1}(A,\mathcal{O}_A(F)) \\ &+ (-1)^{n+1}\dim H^n(M,\mathcal{O}(F-[A])) - (-1)^{n+1}\dim H^n(M,\mathcal{O}(F)) = 0. \end{aligned} \qquad (1)$$

Let $s \in \Gamma(M,\mathcal{O}([A])$ be the section of which the zero locus coinsides with A. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\Gamma(M,\mathcal{O}(F-[A]) & \xrightarrow[\otimes s]{\varphi_1} & \Gamma(M,\mathcal{O}(F)) \\
\downarrow r_1 & & \downarrow r_2 \\
\Gamma(M-E,\mathcal{O}(F-[A]) & \xrightarrow[\otimes s]{\varphi_2} & \Gamma(M-E,\mathcal{O}(F)),
\end{array}$$

where r_i's are restriction mapping and φ_i's are mapping defined by tensoring with the section s. It is easy to see that r_1, r_2 and φ_1 are injections. Since s never vanishes on $M - E$, φ_2 is an isomorphism. Hence

$$0 \longrightarrow \Gamma(M,\mathcal{O}(F))/\varphi_1\Gamma(M,\mathcal{O}(F-[A])) \xrightarrow[\otimes s^{-1}]{}$$
$$\Gamma(M-E,\mathcal{O}(F-[A]))/\Gamma(M,\mathcal{O}(F-[A]) \xrightarrow[\otimes s]{} \Gamma(M-E,\mathcal{O}(F))/\Gamma(M,\mathcal{O}(F))$$
$$\longrightarrow 0.$$

are exact. Therefore

$$\begin{aligned}\dim \Gamma(M,\mathcal{O}(F))/\varphi_1\Gamma(M,\mathcal{O}(F-[A])) &= \dim \Gamma(M-E,\mathcal{O}(F-[A]))/\Gamma(M,\mathcal{O}(F-[A])) \\ &\quad - \dim \Gamma(M-E,\mathcal{O}(F))/\Gamma(M,\mathcal{O}(F)). \end{aligned} \tag{2}$$

By (1) and (2) we obtain

$$\chi(M,\mathcal{O}(F-[A])) - \chi(M,\mathcal{O}(F)) = \chi_A(A,\mathcal{O}_A(F)).$$

Let $\overline{f}$ be the topological "extension" of f to N as in Lemma 1.2. Then

$$T(N,\overline{f}) - T(N,\overline{f}-[A]) = T(A,F|_A) = \chi_A(A,\mathcal{O}_A(F)).$$

Hence

$$\chi(M,\mathcal{O}(F-[A])) + T(N,\overline{f}-[A]) = \chi(M,\mathcal{O}(F)) + T(N,\overline{f}).$$

This shows that

$$\chi(M,\mathcal{O}(F-[D])) + T(N,\overline{f}-[D]) = \chi(M,\mathcal{O}(F)) + T(N,\overline{f})$$

for any integral divisor D on M.

Therefore we obtain

$$\chi(M,\mathcal{O}(F+[D])) - \chi(M,\mathcal{O}(F)) = T(N,\overline{f}) - T(N,\overline{f}+D).$$

Remark. $D\cdot\overline{f} = D\cdot f$.

Corollary 1.5.

$$\chi(M,\mathcal{O}([D])) = -\frac{d^n}{n!} + \frac{d^{n-1}}{(n-1)!}T_1(c_1) + \cdots + dT_{n-1}(c_1,\ldots,c_{n-1}) - \sum_{q=1}^{n-1}(-1)^q \dim H^q(M,\mathcal{O}).$$

Proof. Since $\Gamma(M-E,\mathcal{O})/\Gamma(M,\mathcal{O}) = 0$, we have $\chi(M,\mathcal{O}) = \sum_{q=1}^{n-1}(-1)^{q-1}\dim H^q(M,\mathcal{O})$. Hence we obtain the corollary by (*).

Remark. If D is irreducible, then

$$\chi(M,\mathcal{O}([D])) = -T(D,[D]|_D) - \sum_{q=1}^{n-1}(-1)^q \dim H^q(M,\mathcal{O}).$$

The explicit formulae on generalized Kato's Riemann-Roch theorem for surface singularities and three-fold singularities are summarised.

1. The dimension two case.

Corollary 1.6 (Kato [Ka]). If (X,x) is a surface singularity, then

$$\chi(M,\mathcal{O}(F+[D])) - \chi(M,\mathcal{O}(F)) = \frac{1}{2}(KD - D^2) - FD, \qquad (**)$$

$$\chi(M,\mathcal{O}([D])) = \frac{1}{2}(KD - D^2) + \dim H^1(M,\mathcal{O}).$$

2. The dimension three case.

Lemma 1.7. Let $D = \sum \lambda_i D_i$ be the decomposition of D into its irreducible components. Denote by N_{D_i} the normal bundle of D_i in $M \subset N$. Assume moreover that $\dim_{\mathbb{C}}(X,x) = 3$. Then

$$D \cdot c_2(N) = \sum \lambda_i \{c_2(D_i) + c_1(D_i)c_1(N_{D_i})\},$$

i.e., $D \cdot c_2(N)$ is independent of the compactification N of M.

Proof. Because of the exact sequence

$$0 \longrightarrow T_{D_i} \longrightarrow T_N|_{D_i} \longrightarrow N_{D_i} \longrightarrow 0,$$

we have

$$D_i \cdot c_2(N) = c_2(D_i) + c_1(D_i)c_1(N_{D_i}).$$

Hence we obtain the desired result by linearlity of the intersection pairing.

Corollary 1.8. Let the notation be as above. If (X,x) is a normal isolated singularity of dimension 3, then

$$\chi(M,\mathcal{O}(F+[D])) - \chi(M,\mathcal{O}(F)) = -\frac{1}{6}D^3 + \frac{1}{4}D^2K - \frac{1}{12}D(c_2 + K^2) + \frac{1}{2}FD(K - F - D), \quad (***)$$

and

$$\chi(M,\mathcal{O}([D])) = -\frac{1}{6}D^3 + \frac{1}{4}D^2K - \frac{1}{12}D(c_2 + K^2) + \dim H^1(M,\mathcal{O}) - \dim H^2(M,\mathcal{O}),$$

where c_2 is the second Chern class of a compactification of an affine model of (X,x).

§ 2. Geometric genus $p_g(X,x)$

Let (X,x) be a normal n-dimensional isolated singularity. The geometric genus $p_g(X,x)$ is defined to be the dimension of $\dim_{\mathbb{C}} (R^{n-1}\pi_*\mathcal{O}_M)_x$ where $\pi : M \longrightarrow X$ is a resolution of the singularity. This $p_g(X,x)$ can be expressed in terms of meromorphic n-forms on M.

Theorem 2.1 (Yau [Y1]). Let (X,x) be a normal n-dimensional isolated singularity. Suppose that x is the only singularity of X and X is a Stein space. Let $\pi : M \longrightarrow X$ be a resolution of the singularity. Then

$$\dim H^{n-1}(M,\mathcal{O}) = \dim \Gamma(M - E,\mathcal{O}(K))/\Gamma(M,\mathcal{O}(K))$$

where $E = \pi^{-1}(\{x\})$.

Definition 2.2. Let (X,x) be a normal isolated singularity. We say (X,x) is a quasi-Gorenstein if there exists a holomorphic n-form ω defined on a deleted neighborhood of x, which is nowhere vanishing on this neighborhood.

Remark. Let (X,x) be a normal isolated singularity whose local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay. If the singularity (X,x) is quasi-Gorenstein, then the local ring $\mathcal{O}_{X,x}$ is Gorenstein.

Assume that (X,x) is a quasi-Gorenstein singularity. Then there exists a nowhere vanishing holomorphic n-form ω defined on $X - \{x\}$. Let K_∞ be the part of the divisor of $\pi^*\omega$ on N which is supported on $N - M$. Then $(\omega) \sim K + K_\infty$. Let k, $k_\infty \in$

$H^2(N,\mathbb{Z})$ be the cohomology class represented by the cycle K, K_∞ respectively.

Let $\{ T_k(c_1,\dots,c_k) \}$ be the multiplicative sequence with characteristic power series $Q(x) = x(1 - e^{-x})^{-1}$ ([Hi]). The polynomial T_k are called Todd polynomial. For small n,

$$T_1 = \frac{1}{2}c_1,$$
$$T_2 = \frac{1}{12}(c_2 + c_1^2),$$
$$T_3 = \frac{1}{24}c_1c_2.$$

Lemma 2.3. Let n be a positive interger, then

$$\sum_{k=0}^{n-1} \frac{(-c_1)^{n-k}}{(n-k)!} T_k(c_1,\dots,c_k) = \{(-1)^n-1\}T_n(c_1,\dots,c_n).$$

Proof.

$$\sum_{k=0}^{n} \frac{(-c_1)^{n-k}}{(n-k)!} T_k(c_1,\dots,c_k) = \kappa_n\left[e^{-c_1}\prod_{i=1}^{n}\frac{\gamma_i}{1-e^{-\gamma_i}}\right]$$

$$= \kappa_n\left[e^{-(\gamma_1+\cdots+\gamma_n)}\prod_{i=1}^{n}\frac{\gamma_i}{1-e^{-\gamma_i}}\right] = \kappa_n\left[\prod_{i=1}^{n}\frac{\gamma_i}{e^{\gamma_i}-1}\right] =$$

$$= \kappa_n\left[\prod_{i=1}^{n}\frac{-\gamma_i}{1-e^{-(-\gamma_i)}}\right] = T_n(-c_1,c_2,\cdots,(-1)^i c_i,\cdots,(-1)^n c_n)$$

$$= (-1)^n T_n(c_1,\dots,c_n).$$

Corollary 2.4 ([Hi], Remark 1, p.14). $T_k(c_1,\dots,c_k)$ is divisible by c_1 for k odd.

Lemma 2.5. $T(N) - T(N,k) = \{1-(-1)^n\}T_n(-k,c_2,\dots,c_n)$.

Proof. $T(N) - T(N,k) = T_n(c_1,\dots,c_n) - \kappa_n\left[e^k \prod_{i=1}^{n} \frac{\gamma_i}{1 - e^{-\gamma_i}}\right]$

$$= T_n(c_1,\dots,c_n) - \sum_{j=0}^{n} \frac{k^{n-j}}{(n-j)!} T_j(c_1,\dots,c_j)$$

$$= -\sum_{j=0}^{n-1} \frac{k^{n-j}}{(n-j)!} T_j(c_1,\dots,c_j) = -\sum_{j=0}^{n-1} \frac{k^{n-j}}{(n-j)!} T_j(-k-k_\infty,\dots,c_j)$$

$$= -\sum_{j=0}^{n-1} \frac{k^{n-j}}{(n-j)!} T_j(-k,\dots,c_j) \qquad \left[\ k\cdot k_\infty = 0\ \right]$$

$$= \{1-(-1)^n\}T_n(-k,\dots,c_n).$$

From this lemma, applying our theorem to the case $D = K$ and $F = 0$, we have the following:

Corollary 2.6. Let (X,x) be a normal isolated singularty of dimension n. If (X,x) is quasi-Gorenstein, then

$$\{1-(-1)^n\}\{p_g(X,x) - T_n(-k,c_2,\dots,c_{n-1})\}$$
$$= h^1(\mathcal{O}_M) - h^2(\mathcal{O}_M) + \cdots + (-1)^{n-1}h^{n-2}(\mathcal{O}_M)$$

where $h^i(\mathcal{O}_M) = \dim H^i(M,\mathcal{O}_M)$.

Proof. $\chi(M,\mathcal{O}(K)) - \chi(M) =$

$$= p_g(X,x) - \{\ h^1(\mathcal{O}_M) - h^2(\mathcal{O}_M) + \cdots + (-1)^n h^{n-1}(\mathcal{O}_M)\ \}$$
$$= \{1-(-1)^n\}p_g(X,x)$$
$$- \{\ h^1(\mathcal{O}_M) - h^2(\mathcal{O}_M) + \cdots + (-1)^{n-1}h^{n-2}(\mathcal{O}_M)\ \}.$$

On the other hand, from Lemma 2.5

$$T(N) - T(N,k) = \{1-(-1)^n\}T_n(-k,c_2,\dots,c_n).$$

Hence we obtain the corollary by Theorem 1.4.

Corollary 2.7. Let (X,x) be a normal isolated singularty of odd dimension. If (X,x) is Gorenstein, then

$$p_g(X,x) = T_n(-k, c_2, \ldots, c_{n-1}).$$

Proof. As is well known, $h^i(\mathcal{O}_M) = 0$ for $1 \le i \le n-1$; [Y2,Theorem 2.6, p.434]. So the conclusion now follows immediately.

Corollary 2.8. Let (X,x) be a normal isolated singularity of dimesnion 3. If (X,x) is quasi-Gorenstein, then

$$2\{\ p_g(X,x) - \frac{-k \cdot c_2}{24}\ \} = h^1(\mathcal{O}_M),$$

i.e., the dimension of the second local cohomology group of $\mathcal{O}_{X,x}$ is even.

Corollary 2.9. Let (X,x) be a normal isolated singularity of dimesnion 4. If (X,x) is quasi-Gorenstein, then

$$h^1(\mathcal{O}_M) = h^2(\mathcal{O}_M).$$

Remark. A quasi-homogeneous cone over a three dimensional abelian variety satisfies the comdition of this Corollary.

We now give a formula for the invariant, which is denoted by $\delta(X,x)$ in [K], of a $\mathbb{Q}$-quasi-Gorenstein singularity (X,x) of dimension 3.

Definition 2.10. (X,x) is said to be $\mathbb{Q}$-quasi-Gorenstein if $mK_X \sim 0$ [i.e., $\mathcal{O}(mK_X) \simeq \mathcal{O}_X$] for some positive integer m. We define the index of a $\mathbb{Q}$-quasi-Gorenstein singularity (X,x) by

$$r(X,x) = \min \{\ m \in \mathbb{N} \mid mK_X \sim 0\ \}.$$

Let (X,x) be a normal isolated singularity and $\pi : M \longrightarrow X$ be a resolution. Then (X,x) is said to be canonical if $K_M - \pi^* K_X$ is effective.

Let (X,x) be a germ of a normal isolated singularity of dimension 3. Suppose that (X,x) is $\mathbb{Q}$-quasi-Gorenstein. Let $\pi : M \longrightarrow X$ be a good resolution of the singularity. We write $K_M = \pi^* K_X + \Delta$ and $\Delta = \sum_j a_j D_j$ where the D_j are exceptional divisors of π. Let r be the index of the singularity (X,x). Then

$$rK_M \cdot c_2(M) = r(\Delta \cdot c_2(N)) = r\Big[\sum_j a_j \{c_2(D_j) + c_1(D_j) c_1(N_{D_j})\Big].$$

It is well known that $(\Delta \cdot c_2(M))$ does not depend on the choice of the resolution π. This was proved by [K. Lemma 2.1, p.541] under the condition that (X,x) is terminal.

We now apply Corollary 1.8 to the case of $\mathbb{Q}$-quasi-Gorenstein singularities. Let r be the index of the singularity (X,x). Substituting rK_M for D in the formula of Corollary 1.8, we obtain

Theorem 2.11. Let (X,x) be a normal 3-dimensional isolated singularity, and let $\pi : M \longrightarrow X$ be a resolution. Suppose that (X,x) is $\mathbb{Q}$-quasi-Gorenstein with index r. Then

$$K_M \cdot c_2(M) = \frac{12}{r}\Big[\chi(M, \mathcal{O}(rK_M)) + \frac{1}{12}(2r^3 - 3r^2 + r) K_M{}^3 - \dim H^1(M,\mathcal{O}) + \dim H^2(M,\mathcal{O})\Big].$$

Corollary 2.12. Let the notation be as above. Assume moreover that (X,x) is canonical. Then

$$K_M \cdot c_2(M) = (2r^2 - 3r + 1)K_M{}^3$$
$$+ \frac{12}{r}\{ \dim H^1(M,\mathcal{O}(rK_M)) - \dim H^2(M,\mathcal{O}(rK_M)) \}.$$

Example 2.13. Let G act on $\mathbb{C}^3$ with coordinates $\{z_1, z_2, z_3\}$ by $g(z_k) = -z_k$, where g is a generator of G. Consider the quotient singularity $(\mathbb{C}^3/G, 0)$. Then, by the result of Kawamata [K, Lemma 2.2, p.542]

$$\delta = 2 - \frac{1}{2}.$$

On the other hand, we have the resolution of the singualrty as follows: M is the total space of the line bundle over $\mathbb{P}^2$ associated to the $-2H$, where H is a hyperplane in $\mathbb{P}^2$. Therefore $2K_M \sim \mathbb{P}^2$, where $\mathbb{P}^2$ is identified with the zero section. Thus $K_M^3 = \frac{1}{2}$, because $(\mathbb{P}^2)^3 = (-2H)^2 = 4$. If $i \geq 1$,

$$H^i(M,\mathcal{O}(2K_M)) = \bigoplus_{k\geq 0} H^i(\mathbb{P}^2,\mathcal{O}(2K_{\mathbb{P}^2} + (k+2)(2H)))$$
$$= \bigoplus_{k\geq 0} H^i(\mathbb{P}^2,\mathcal{O}((2k-2)H)).$$

For all $k \geq 0$, $(2k-2)H > K_{\mathbb{P}^2}$. Then $H^i(M,\mathcal{O}(2K_M)) = 0$ for $i = 1, 2$ by the Kodaira vanishing theorem.

§ 3. Examples

In this section we give a few examples of normal isolated singularities, whose plurigenera will be calculated in the next sections.

Example 3.1. Let (X,x) be a normal surface singularity defined by the polynomial $x^8 + y^8 + z^8 + (xyz)^2$. We first describe the minimal resolution $\pi : M \longrightarrow X$ of the singularity. One obtains $E = \pi^{-1}(0) = A \cup B \cup C$, a union of 3 curves of genus 3. Moreover $A^2 = B^2 = C^2 = E^2 = -6$, $AB = BC = CA = 2$ and $K_M \sim -5E$.

Example 3.2. Let S be a nonsingular projective surface and let F be an ample line bundle on S. Then

$$X := \operatorname{Spec} \bigoplus_{k=0}^{\infty} \Gamma(S, \mathcal{O}(kF))$$

is a quasi-homogeneous cone of dimension 3 which is smooth outside its vertex. If M denotes the total space of F^{-1}, then the natural map $\pi : M \longrightarrow X$ is a resolution of X whose exceptional locus is just the zero section of M. If we identify the zero section with S, then its normal bundle in M equals F^{-1}.

Assume moreover $K_S = rF$ for some positive integer r. Then $K_M = -(r+1)S$, i.e., (X,x) is quasi-Gorenstein. Our theorem

$$2\{p_g(X,x) - \frac{1}{24}(-K_M)c_2(M)\} = h^1(M,\mathcal{O})$$

turns into

$$2p_g(X,x) - h^1(M,\mathcal{O}) = \frac{r+1}{12}\{c_2(S) + c_1(S)c_1(N_S)\},$$

i.e.,

$$2p_g(X,x) - h^1(M,\mathcal{O}) = \frac{r+1}{12}\{c_2(S) + c_1(S)c_1(F)\}.$$

On the other hand,

$$h^1(M,\mathcal{O}) = \dim_{\mathbb{C}} H^1(M,\mathcal{O}) = \sum_{k=0}^{\infty} h^1(S,\mathcal{O}(kF)),$$

$$p_g(X,x) = \dim_{\mathbb{C}} H^2(M,\mathcal{O}) = \sum_{k=0}^{\infty} h^2(S,\mathcal{O}(kF)).$$

The canonical line bundel K_S is ample. Then $h^i(S,\mathcal{O}(kF)) = 0$ for $k \geq r+1$ and $i \geq 1$ by the Kodaira vanishing theorem and hence

$$h^1(M,\mathcal{O}) = h^1(\mathcal{O}) + h^1(F) + \cdots + h^1(rF),$$

$$p_g(X,x) = h^2(\mathcal{O}) + h^2(F) + \cdots + h^2(rF).$$

The Serre duality says

$$p_g(X,x) = h^0(\mathcal{O}) + h^0(F) + \cdots + h^0(rF).$$

Thus

$$\sum_{k=0}^{r} \{h^0(kF) - h^1(kF) + h^2(kF)\} = \frac{1}{12}\{c_2(S) + c_1(S)c_1(F)\}\times(r+1).$$

This is proved easily by the Riemann-Roch Theorem for surfecs:

$$\{h^0(kF) - h^1(kF) + h^2(kF)\} = \frac{1}{2}\{(kf)^2 + \frac{1}{2}(kf)c_1 + \frac{1}{12}(c_2 + c_1^2)\}.$$

Example 3.3. Consider the hypersurface singularity (X,x)

$$x^{10} + y^{10} + z^{10} + w^{10} - (xyzw)^2 = 0.$$

Let B be the algebraic surface in $\mathbb{P}^3$ by the equation $\xi_0^{10} + \xi_1^{10} + \xi_2^{10} + \xi_3^{10} = 0$. In particular, we denote the zero set of ξ_i by H_i, for $i = 0, 1, 2, 3$. Let $\pi : V \longrightarrow \mathbb{P}^3$ be the double covering of $\mathbb{P}^3$ with branch locus B. Let $R = \pi^{-1}(B)$, $S_j = \pi^{-1}(H_j)$ for $j = 0, 1, 2$ and 3. Then S_j is a double covering of $\mathbb{P}^2$ with branch locus $B \cap H_i$. Blow up V at each point of $S_i \cap S_j \cap R$ for all i and j. We denote the blowing-up by $\sigma_0 : V_0 \longrightarrow V$.

Let ${}_kE_{ij}^{(0)}$ be the exceptional sets, which are isomorphic to $\mathbb{P}^2$. We denote the proper transforms of S_i and R by $S_i^{(0)}$ and $R^{(0)}$. Set $r_i^{(0)} = S_i^{(0)} \cap R^{(0)}$.

We denote the monoidal transformation of V_0 along $r_i^{(0)}$'s by $\sigma_1 : V_1 \longrightarrow V_0$. Let $S_i^{(1)}$, $R^{(1)}$, ${}_kE_{ij}^{(1)}$ be the proper transform of $S_i^{(0)}$, $R^{(0)}$, ${}_kE_{ij}^{(0)}$. Let $F_i^{(1)}$ be the exceptoional set. Set $r_i^{(1)} = F_i^{(1)} \cap S_i^{(1)}$ and ${}_kh_{ij}^{(1)} = {}_kE_{ij}^{(1)} \cap R^{(1)}$.

We denote the monoidal transformation of V_1 along ${}_kh_{ij}^{(1)}$'s by $\sigma_2 : V_2 \longrightarrow V_1$. Let $S_i^{(2)}$, $R^{(2)}$, ${}_kE_{ij}^{(2)}$, $F_i^{(2)}$ be the proper transform of $S_i^{(1)}$, $R^{(1)}$, ${}_kE_{ij}^{(1)}$, $F_i^{(1)}$.

Let ${}_kG_{ij}^{(2)}$ be the exceptional sets. Let $r_i^{(2)}$ be the inverse image of $r_i^{(1)}$, i.e., $r_i^{(2)} = S_i^{(2)} \cap F_i^{(2)}$.

$\dfrac{\xi_i^2 \xi_\alpha \xi_\beta \xi_\gamma}{\sqrt{\xi_0^{10}+\xi_1^{10}+\xi_2^{10}+\xi_3^{10}}}$ can be considered as a meromorphic function on V_2 and is holomorphic off $R^{(2)}$.

The following functions on $M = V_2 - R^{(2)}$

$$\varphi_0 = \frac{\xi_0^2 \xi_1 \xi_2 \xi_3}{\sqrt{\xi_0^{10}+\xi_1^{10}+\xi_2^{10}+\xi_3^{10}}}, \quad \varphi_1 = \frac{\xi_0 \xi_1^2 \xi_2 \xi_3}{\sqrt{\xi_0^{10}+\xi_1^{10}+\xi_2^{10}+\xi_3^{10}}},$$

$$\varphi_2 = \frac{\xi_0 \xi_1 \xi_2^2 \xi_3}{\sqrt{\xi_0^{10}+\xi_1^{10}+\xi_2^{10}+\xi_3^{10}}}, \quad \varphi_3 = \frac{\xi_0 \xi_1 \xi_2 \xi_3^2}{\sqrt{\xi_0^{10}+\xi_1^{10}+\xi_2^{10}+\xi_3^{10}}}$$

shows that $E = \sum_{i=0}^{3} S_i^{(2)} + \sum_{0 \le i < j \le 3} \sum_{k=1}^{10} {}_kE_{ij}^{(2)}$ is excptional in M.

$\rho = (\varphi_0, \varphi_1, \varphi_2, \varphi_3) : M \longrightarrow \mathbb{C}^4$ is biholomorphic off E.

ρ maps M to

$$X = \{ (x_0, x_1, x_2, x_3) \in \mathbb{C}^4 \mid f(x) = 0 \},$$

where $f(x) = x_0^{10} + x_1^{10} + x_2^{10} + x_3^{10} - (x_0 x_1 x_2 x_3)^2$. Then

$$\omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{f_{x_0}} = -\frac{dx_0 \wedge dx_2 \wedge dx_3}{f_{x_1}} = \frac{dx_0 \wedge dx_1 \wedge dx_3}{f_{x_2}} = -\frac{dx_0 \wedge dx_1 \wedge dx_2}{f_{x_3}}$$

is a nowhere vanishing holomorphic 3-form on $X - \{0\}$. $\rho^*(\omega)$ extends to a meromorphic 3-form on M with pole set contained in $E = \rho^{-1}(0)$. K_M, the divisor of ω and also called the canonical divisor, is described as follows:

$$(*) \qquad K_M = -6(S_0^{(2)} + S_1^{(2)} + S_2^{(2)} + S_3^{(3)}).$$

π is the double covering of $\mathbb{P}^3$ with branch locus $R \cap H_i$, so the Euler number of S_i is 76 and $K_V \sim \pi^* K_{\mathbb{P}^3} + R$. Hence $c_2(S_i^{(2)}) = 106$ and

$$K_{V_2} \sim -S_0^{(2)} - S_1^{(2)} - S_2^{(2)} - S_3^{(2)} + F_0^{(2)} + F_1^{(2)} + F_2^{(2)} + F_3^{(2)} + \sum_{0 \le i < j \le 3} \left(\sum_{k=1}^{10} {}_kE_{ij}^{(2)} + 3 \sum_{k=1}^{10} {}_kG_{ij}^{(2)} \right) + R^{(2)},$$

where $f_{ij} = S_i^{(2)} \cap S_j^{(2)}$.

Using the adjunction formula, we have

$$K_{S_i^{(2)}} \sim -f_{i\alpha} - f_{i\beta} - f_{i\gamma} + r_i^{(2)} + \sum_{k=1}^{10} \left({}_ke_{i\alpha,i}^{(2)} + {}_ke_{i\beta,i}^{(2)} + {}_ke_{i\gamma,i}^{(2)} \right).$$

For a meormorphic function f, we denote by (f) the divisor of f. Then

$$\left(\frac{\xi_i^2\xi_\alpha\xi_\beta\xi_\gamma}{\sqrt{\xi_0^{10}+\xi_1^{10}+\xi_2^{10}+\xi_3^{10}}}\right) = 2S_i^{(2)} + S_\alpha^{(2)} + S_\beta^{(2)} + S_\gamma^{(2)} + + 2\left(\sum_{k=1}^{10} {}_kE_{i\alpha}^{(2)} + {}_kE_{i\beta}^{(2)} + {}_kE_{i\gamma}^{(2)}\right) + F_i^{(2)} + + \left(\sum_{k=1}^{10} {}_kE_{\alpha\beta}^{(2)} + {}_kE_{\beta\gamma}^{(2)} + {}_kE_{\gamma\alpha}^{(2)}\right) + + \sum_{k=1}^{10}\left({}_kG_{i\alpha}^{(2)} + {}_kG_{i\beta}^{(2)} + {}_kG_{i\gamma}^{(2)}\right) \quad \text{on } M.$$

Hence

$$-2S_i^{(2)} \sim S_\alpha^{(2)} + S_\beta^{(2)} + S_\gamma^{(2)} + + 2\left(\sum_{k=1}^{10} {}_kE_{i\alpha}^{(2)} + {}_kE_{i\beta}^{(2)} + {}_kE_{i\gamma}^{(2)}\right) + F_i^{(2)} + + \left(\sum_{k=1}^{10} {}_kE_{\alpha\beta}^{(2)} + {}_kE_{\beta\gamma}^{(2)} + {}_kE_{\gamma\alpha}^{(2)}\right) + + \sum_{k=1}^{10}\left({}_kG_{i\alpha}^{(2)} + {}_kG_{i\beta}^{(2)} + {}_kG_{i\gamma}^{(2)}\right) \quad \text{on } M = V_2 - R^{(2)}.$$

Thus

$$N_{S_i^{(2)}} = [S_i^{(2)}]\Big|_{S_i^{(2)}} = -\frac{1}{2}\Big[S_\alpha^{(2)} + S_\beta^{(2)} + S_\gamma^{(2)} + + 2\sum_{k=1}^{10}\left({}_kE_{i\alpha}^{(2)} + {}_kE_{i\beta}^{(2)} + {}_kE_{i\gamma}^{(2)}\right) + F_i^{(2)}\Big]\Big|_{S_i^{(2)}} \quad \text{on } M.$$

i.e.,

$$-2\left(N_{S_i^{(2)}}\right) = \Big[f_{i\alpha} + f_{i\beta} + f_{i\gamma} + r_i^{(2)} + + 2\sum_{k=1}^{10}\left({}_ke_{i\alpha,i}^{(2)} + {}_ke_{i\beta,i}^{(2)} + {}_ke_{i\gamma,i}^{(2)}\right)\Big],$$

where $\{\alpha, \beta, \gamma, i\} = \{1, 2, 3, 4\}$ and ${}_kE_{ij}^{(2)} \cap S_i^{(2)} =$

${}_k e^{(2)}_{ij,i}$. Since $f^2_{ij} = -8$, $f_{ij}\cdot f_{jk} = 2$ and $\{r^{(2)}_i\}^2 = 20$,

$$c_1(S^{(2)}_i)c_1(N_{S^{(2)}_i}) = 16.$$

Therefore

$$S^{(2)}_i \cdot c_2(V_2) = c_2(S^{(2)}_i) + c_1(S^{(2)}_i)c_1(N_{S^{(2)}_i}) = 122.$$

$S^{(2)}_i$'s are isomorphic one another. So from (*),

$$(-K_M)\cdot c_2(V_2) = 122\times 4\times 6.$$

Thus

$$p_g(X,x) = 122.$$

§ 4. δ_m

We now calculate plurigenera $\{ \delta_m \}$ of certain hypersurfec singularities by means of generalized Kato's Riemann-Roch Theorem.

We need to recall a few preliminaries related to the concept of plurigenera of normal isolated singularities.

Let (X,x) be a normal isolated singularity of an n-dimensional analytic space X. Let V be a (sufficiently small) Stein neighborhood of x and let K be the canonical line bundle of $V - \{x\}$. An elment of $\Gamma(X-\{x\},\mathcal{O}(mK))$ is considered as a holomorphic m-ple n-form. Let ω be a holomorpic m-ple n-form on $X - \{x\}$. We write ω as

$$\omega = \phi(z)(dz_1\wedge dz_2\wedge\cdots\wedge dz_n)^m,$$

using local coordinates $(z_1, z_2, \ldots, Z_n)$. We associate with ω the continuous local (n,n)-form $(\omega\wedge\bar{\omega})^{1/m}$ given by

$$|\phi(z)|^{2/m}(\sqrt{-1}/2)^n dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2\wedge\cdots\wedge dz_n\wedge d\bar{z}_n \ .$$

Definition 4.1. ω is called integrable ($L^{2/m}$-integrable) if

$$\int_{W - \{x\}} (\omega\wedge\bar{\omega})^{1/m} < \infty$$

for any sufficiently small relatively compact neighborhood W of x in X.

Let $L^{2/m}(V - \{x\})$ be the set of all integrable holomorphic m-ple n-form on $V - \{x\}$, which is a linear subspace of $\Gamma(V - \{x\}, \mathcal{O}(mK))$. Then $\Gamma(V-\{x\},\mathcal{O}(mK))/L^{2/m}(V-\{x\})$ is a finite dimensional vector space.

Definition 4.2. The m-th plurigenus , m being a positive integer, of a normal isolated singularity (X,x) is

$$\delta_m(X,x) = \dim \Gamma(V-\{x\},\mathcal{O}(mK))/L^{2/m}(V-\{x\}).$$

These integers $\{\delta_m\}$ are determined independently of the choice of the Stein neighborhood.

Let $\pi : M \longrightarrow X$ be a resolution of (X,x) and $U = \pi^{-1}(V)$ and $E = \pi^{-1}(\{x\})$. By Sakai [S, Theorem 2.1, p.243],

$$L^{2/m}(V - \{x\}) \simeq L^{2/m}(U - E) = \Gamma(U,\mathcal{O}(mK + (m-1)E))$$

if the exceptional set is a divisor which has at most normal crossings. Thus

$$\delta_m(X,x) = \Gamma(U-E,\mathcal{O}(mK))/\ \Gamma(U,\mathcal{O}(mK + (m-1)E)).$$

This formula provides a practical means to compute δ_m in many cases. Further details can be found in [Wa].

The following theorem due to Kawamata and Viehweg [KMM] is very useful.

Vanishing Theorem. Let F be a line bundle on M which satisfies $F\cdot C \geq 0$ for any curve in the exceptional set. Then $H^i(M,\mathcal{O}(K+F)) = 0$ for $i \geq 1$.

There is an interesting case in which the δ_m can be calculated by means of generalized Kato's Riemann-Roch theorem and Vanishing Theorem:

Suppose that (X,x) is quasi-Gorenstein and $K + E$ is nef. Then

$$\chi(M,\mathcal{O}(mK+(m-1)E)) = \delta_m(X,x) \quad \text{for} \quad m \geq 1.$$

$$\chi(M,\mathcal{O}(mK+(m-1)E)) = -\frac{(K+E)^n}{n!}m^n + \frac{(K+E)^{n-1}(K+2E)}{2(n-1)!}m^{n-1} + O(m^{n-2}).$$

$$\lim_{m\to\infty} \frac{\delta_m}{m^n} = -\frac{(K+E)^n}{n!} \neq 0.$$

The above hypothesis is for instance satisfied if (X,x) is a hypersurface singularities defined by the equation $x^8 + y^8 + z^8 - (xyz)^2 = 0$ or $x^{10} + y^{10} + z^{10} + w^{10} - (xyzw)^2 = 0$.

Corollary 4.3. Let (X,x) be a quasi-Gorenstein singularity of dimension 2. If $K + E$ is nef, then

$$\delta_m = \chi(M,\mathcal{O}(mK+(m-1)E))$$
$$= -\frac{1}{2}(K+E)^2m^2 + \frac{1}{2}(K+E)(K+2E)m - \frac{1}{2}(E^2+EK) + \dim H^1(M,\mathcal{O}).$$

By Corollary 4.3 we have the following:

Proposition 4.4. Let (x,x) be a normal surface singularity defined by the polynomial $x^8 + y^8 + z^8 - (xyz)^2$. Then, from the data of Example 3.1

$$\delta_m(X,x) = 48m^2 - 36m + 20.$$

Corollary 4.5. Let (X,x) be a quasi-Gorenstein singularity of dimension 3. If $K + E$ is nef, then

$$\delta_m(X,x) = \chi(M,\mathcal{O}(mK+(m-1)E))$$
$$= -\frac{(K+E)^3}{6}m^3 + \frac{(K+E)^2(K+2E)}{4}m^2 - \frac{1}{12}(K+E)(6E^2+6EK+K^2+c_2)m$$
$$+ \{\frac{1}{6}E^3 + \frac{1}{4}E^2K + \frac{1}{12}E(c_2 + K^2)\} + \dim H^1(M,\mathcal{O}) - \dim H^2(M,\mathcal{O}).$$

Hence we obtain the following proposition.

Proposition 4.6. If (X,x) is defined by the polynomial $x^{10} + y^{10} + z^{10} + w^{10} - (xyzw)^2 = 0$, then, from the date of Example 3.3,

$$\delta_m(X,x) = \frac{500}{3}m^3 - 200m^2 + \frac{670}{3}m - 68.$$

§ 5. d_m

The following integer is defined by Ishii [I]:

$$d_m(X,x) := \dim_{\mathbb{C}} \frac{\Gamma(M,\mathcal{O}(mK+mE))}{\Gamma(M,\mathcal{O}(mK+(m-1)E))} \quad (m \geq 1).$$

This integer is determined independently of the choice of the strongly pseudoconvex neighborhoods. Hence d_m can be seen as an invariant attached to the singularity. Ishii considered the asymptotic behvior of d_m when $m \longrightarrow +\infty$. We calculate the value

$$d = \lim_{m \to +\infty} \frac{d_m}{m^{n-1}}$$

for certain singularities.

Let

$$\varepsilon_m(X,x) := \dim_{\mathbb{C}} \frac{\Gamma(M-E,\mathcal{O}(mK))}{\Gamma(M,\mathcal{O}(mK+mE))} \quad (\text{ for } m \geq 1)$$

be the third plurigenera of normal isolated singularities. We consider the following exact sequence:

$$0 \longrightarrow \frac{\Gamma(M,\mathcal{O}(mK+mE))}{\Gamma(M,\mathcal{O}(mK+(m-1)E))} \longrightarrow \frac{\Gamma(M-E,\mathcal{O}(mK))}{\Gamma(M,\mathcal{O}(mK+(m-1)E))} \longrightarrow \frac{\Gamma(M-E,\mathcal{O}(mK))}{\Gamma(M,\mathcal{O}(mK+mE))} \longrightarrow 0 .$$

Then we have $d_m = \delta_m - \varepsilon_m$.

Lemma 5.1. If $K + E$ and $K + 2E$ are nef, then

$$\chi(M,\mathcal{O}(mK+mE)) = \dim_{\mathbb{C}} \frac{\Gamma(M-E,\mathcal{O}(mK))}{\Gamma(M,\mathcal{O}(mK+mE))} = \varepsilon_m \quad \text{for} \quad m \geq 2.$$

Proof. $K + (m - 2)(K + E) + (K + 2E)$ is nef, so the conclusion now follows immediately from Vanishing Theorem.

Consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}(K) \longrightarrow \mathcal{O}(K+E) \longrightarrow \mathcal{O}_E(K) \longrightarrow 0.$$

Then, by Grauert-Riemenschneider vanishing theorem, the sequence

$$0 \longrightarrow \Gamma(M,\mathcal{O}(K)) \longrightarrow \Gamma(M,\mathcal{O}(K+E)) \longrightarrow \Gamma(E,\mathcal{O}_E(K)) \longrightarrow 0$$

is exact. Then

$$d_1(X,x) = \dim \Gamma(E,\mathcal{O}_E(K)) = \text{geometric genus of } E.$$

We now give an interesting case in which the d_m can be calculated by means of generalized Kato's Riemann-Roch theorem and Vanishing Theorem:

Suppose $K + E$ and $K + 2E$ are nef. Then

$$\chi(M,\mathcal{O}(mK+(m-1)E)) - \chi(M,\mathcal{O}(mK+mE)) = d_m \quad \text{for} \quad m \geq 2.$$

$$\chi(M,\mathcal{O}(mK+(m-1)E)) - \chi(M,\mathcal{O}(mK+mE)) = \frac{(K+E)^{n-1}E}{(n-1)!}m^{n-1} + O(m^{n-2}),$$

$$\lim_{m \to +\infty} \frac{d_m}{m^{n-1}} = \frac{(K+E)^{n-1}E}{(n-1)!} \neq 0.$$

We conclude this paper with explicit d_m-formulae for the singularities of surfaces and three-folds.

Lemma 5.2. Let the notaion be as above. Then

$$\chi(E,\mathcal{O}(K_E)) = \chi(M,\mathcal{O}(K)) - \chi(M,\mathcal{O}(K+E)).$$

Suppose that $K + E$ and $K + 2E$ are nef. Then, in the three-fold case, the above formula (***) and Serre duality give

$$d_1 = \frac{1}{6}E^3 + \frac{1}{4}E^2K + \frac{1}{12}E(c_2 + K^2) + h^1(E,\mathcal{O}) - 1,$$

$$d_m(X,x) = \frac{(K+E)^2E}{2}m^2 - \frac{1}{2}(K+E)^2Em + \{\frac{1}{6}E^3 + \frac{1}{4}E^2K + \frac{1}{12}E(c_2 + K^2)\} \quad \text{for} \quad m \geq 2.$$

Example 5.3. Consider the normal isolated singularity defined by the equation $x^{10} + y^{10} + z^{10} + w^{10} - (xyzw)^2 = 0$. We already know that $E^3 = 8$, $K = -6E$, $Ec_2 = 488$ and $h^1(E,\mathcal{O}) = 0$, because (X,x) is Cohen-Macaulay. Since $-E$ is nef, $K + E$ and $K + 2E$ are also nef. Therefore

$$d_1 = 54 - 1 = 53.$$

$$d_m = 100m^2 - 100m + 54 \quad \text{for} \quad m \geq 2.$$

Remark. In our case $\mathcal{O}(-E)$ is locally principal, so E^3 is equal to the multiplicity of the singularity (c.f. [W, Theorem 5. 1, p.444]). Therefore $E^3 = 8$. This was pointd out to the auther by Tomari.

Next consider the 2 dimensional case. Then Formula (**) and Serre duality give

$$d_1(X,x) = \frac{1}{2}(K+E)E + h^1(E,\mathcal{O}_E(K)) = \frac{1}{2}(K+E)E + 1,$$

$$d_m(X,x) = (K+E)Em - \frac{1}{2}(K+E)E \quad \text{for} \quad m \geq 2.$$

Example 5.4. Consider the singularity defined by the equation $x^8 + y^8 + z^8 + (xyz)^2 = 0$. We already know that $E^2 = -6$ and $K = -5E$. Therefore

$$d_1 = 12 + 1,$$

$$d_m = 24m - 12 \quad m \geq 2.$$

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On a free resolution of the module of logarithmic forms of a generic arrangement

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1 Introduction

1.1 The setup of this paper

Let V be an ℓ-dimensional vector space over a field $\mathbb{K}$. We assume that the characteristic of $\mathbb{K}$ is zero. (For the positive characteristic case, see 4.6.) Let $\mathcal{A}$ be a generic arrangement of n hyperplanes: $\mathcal{A}$ is a finite family of one-codimensional subspaces of V satisfying

1. $n = \#\mathcal{A} > \ell \geq 3$,
2. every ℓ hyperplanes of $\mathcal{A}$ intersect only at the origin.

Let S denote the symmetric algebra $S(V^*)$ of the dual space V^* of V. Then S is considered as the $\mathbb{K}$-algebra of all polynomial functions on V. Let $0 \leq q \leq \ell$. Let $\Omega^q = \Omega^q_S$ denote the module of all regular q-forms on V. Then each Ω^q is a free S-module of rank $\binom{\ell}{q}$. For each $H \in \mathcal{A}$ choose $\alpha_H \in V^*$ such that $\ker(\alpha_H) = H$. Let

$$Q = Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H \in S.$$

Define

$$\begin{aligned}\Omega^q(\mathcal{A}) \ = \{\omega \mid \ & \omega \text{ is a (global) rational } q\text{-form on } V \text{ such that} \\ & \text{both } Q\omega \text{ and } Q(d\omega) \text{ lie in } \Omega^q_S\},\end{aligned}$$

which is called the **module of logarithmic q-forms with pole along $\mathcal{A}$.** Introduce a grading into $\Omega^q(\mathcal{A})$ as usual: for example

$$\deg(f dy_1 \wedge \cdots \wedge dy_q) = \deg f + q,$$

where f is a homogeneous element in S and $y_i \in V^*$ $(1 \leq i \leq q)$. Then $\Omega^q(\mathcal{A})$ is a finitely generated graded S-module.

1.2 The aim

Let

$$J = J(Q) := (\partial Q/\partial x_1, \ldots, \partial Q/\partial x_\ell)S$$

be the Jacobian ideal of Q. At the NSF-CBMS regional meeting research conference on " Arrangements of hyperplanes" in 1988, Yuzvinsky posed the conjecture that the depth (as an S-module) of the factor ring S/J is equal to zero:

$$\mathrm{depth}_S S/J = 0.$$

This conjecture is equivalent to the conjecture that the homological dimension of $\Omega^{\ell-1}(\mathcal{A})$ is equal to $\ell - 2$. In this paper we prove this conjecture. Actually we shall prove:

$$\begin{aligned}\mathrm{hd}_S \Omega^q(\mathcal{A}) &= q \quad (0 \leq q \leq \ell - 2), \\ \mathrm{hd}_S \Omega^{\ell-1}(\mathcal{A}) &= \ell - 2, \\ \mathrm{hd}_S \Omega^\ell(\mathcal{A}) &= 0.\end{aligned}$$

Moreover we shall construct an explicit minimal free resolution of $\Omega^q(\mathcal{A})$ for $0 \leq q \leq \ell$. In particular the free resolution of $\Omega^{\ell-1}(\mathcal{A})$ gives a minimal free resolution of the module $D(\mathcal{A})$ of logarithmic derivations along $\mathcal{A}$ studied in [4] and others. A minimal free resolution of the factor ring S/J is also given.

1.3 Outline

Fix a hyperplane $H_1 \in \mathcal{A}$ for a moment. Let

$$\mathcal{A}' = \mathcal{A} \setminus \{H_1\}, \quad \mathcal{A}'' = \{H \cap H_1 \mid H \in \mathcal{A}'\}.$$

(Then $\mathcal{A}''$ is an arrangement in the $(\ell - 1)$-dimensional vector spce H_1.)

First in Section 2 we shall prove that there exists a short exact sequence

$$0 \to \Omega^q(\mathcal{A}') \to \Omega^q(\mathcal{A}) \to \Omega^{q-1}(\mathcal{A}'') \to 0$$

in 2.3.6. We also show (2.3.4) that $\Omega^q(\mathcal{A})$ is generated as an S-algebra by

$$\{d\alpha/\alpha \mid \alpha \in V^*, \ker \alpha \in \mathcal{A}\}$$

for $0 \leq q \leq \ell - 2$.

In Section 3, we shall first show that the Fitting ideal of $\Omega^1(\mathcal{A})$ has the maximal height ℓ in 3.2.3 . Combining with a result of [1], we can show that

$$\bigwedge^q \Omega^1(\mathcal{A}) \simeq \Omega^q(\mathcal{A})$$

for $0 \leq q \leq \ell - 2$ in 3.3.2. Using this, we shall construct an explicit minimal free resolution of $\Omega^q(\mathcal{A})$ for $0 \leq q \leq \ell - 2$. The resolution is composed of symmetric powers and exterior powers. The resolution of this type was studied by Lebelt in [2], which we actually use. The homological dimension of $\Omega^q(\mathcal{A})$ turns out to be equal to q for $0 \leq q \leq \ell - 2$.

In Section 4 we shall construct a minimal free resolution of $\Omega^q(\mathcal{A})$ for $q = \ell - 1$, which is the only remaining case. Fix $\alpha_1 \in V^*$ with $\ker \alpha_1 = H_1 \in \mathcal{A}$ for a moment. First in 4.2.4 we get an exact sequence

$$\cdots \xrightarrow{\partial} \Omega^q(\mathcal{A}) \xrightarrow{\partial} \Omega^{q+1}(\mathcal{A}) \xrightarrow{\partial} \cdots,$$

where ∂ is defined by

$$\partial(\omega) = (d\alpha_1/\alpha_1) \wedge \omega \quad (\omega \in \Omega^q(\mathcal{A}))$$

for $0 \leq q \leq \ell$. Combining this exact sequence with the free resolutions of $\Omega^q(\mathcal{A})$ $(0 \leq q \leq \ell - 2)$ in Section 3, we shall construct a minimal free resolution of $\Omega^{\ell-1}(\mathcal{A})$ in 4.3.7. Its homological dimension turns out to be equal to $\ell - 2$. The resolution yields a minimal free resolution 4.5.3 of the factor ring S/J of S by the Jacobian ideal $J = J(Q)$. The length of the resolution is ℓ, so we have

$$\mathrm{hd}_S S/J = \ell, \quad \mathrm{depth}_S S/J = 0,$$

as conjectured by Yuzvinsky.

Let $D(\mathcal{A})$ be the module of logarithmic derivations along $\mathcal{A}$ (defined in 4.4). Since

$$\Omega^{\ell-1}(\mathcal{A}) \simeq D(\mathcal{A})$$

as S-modules, we have a minimal free resolution of $D(\mathcal{A})$ in 4.4.2. The characteristic sequence in [6] of a generic arrangement $\mathcal{A}$ is proved to be:

$$(1, \underbrace{n-\ell+1, \ldots, n-\ell+1}_{\binom{n-1}{\ell-2}}; \ldots; \quad \underbrace{n-\ell+k, \ldots, n-\ell+k}_{\binom{n-\ell+k-2}{k-1}\binom{n-1}{\ell-1-k}} \quad ; \ldots$$
$$; \underbrace{n-1, \ldots, n-1}_{\binom{n-3}{\ell-2}})$$

in 4.4.4.

2 A short exact sequence and generators for $\Omega^q(\mathcal{A})$

2.1 The setup

Let $V, \ell, \mathbf{K}, \mathcal{A}, n, S, \Omega^q, Q, \Omega^q(\mathcal{A})$ be as in 1.1. Fix $H_1 \in \mathcal{A}$ in this section. Let

$$\mathcal{A}' = \mathcal{A} \setminus \{H_1\}, \qquad \mathcal{A}'' = \{H \cap H_1 \mid H \in \mathcal{A}'\}.$$

Then $\mathcal{A}'$ is an arrangement in V and is called the **deletion** of $\mathcal{A}$. Note that $\mathcal{A}'$ is also generic if $\#\mathcal{A} = n > \ell + 1$. The arrangement $\mathcal{A}''$, called the **restriction** of $\mathcal{A}$ to H_1, is always a generic arrangement in H_1 unless $\ell = 3$. Choose $\alpha_1 \in V^*$ with $\ker \alpha_1 = H_1$. Let $Q' = Q/\alpha_1$. Then Q' defines $\mathcal{A}'$. Let $x_1, \ldots, x_\ell$ be a basis for V^*.

2.2 The residue map

The results in this subsection are true for any nonempty (may not be generic) arrangement $\mathcal{A}$.

Lemma 2.2.1 *For any $\omega \in \Omega^q(\mathcal{A})$, there exist a rational $(q-1)$-form ω' and a rational q-form ω'' such that*

1. $\omega = \omega' \wedge (d\alpha_1/\alpha_1) + \omega''$,
2. *$Q'\omega'$ and $Q'\omega''$ are both regular (no pole).*

Proof. We can assume that $\alpha_1 = x_1$. One may (uniquely) choose ω' and ω'' such that

1. $\omega = \omega' \wedge (dx_1/x_1) + \omega''$,

2. neither ω' nor ω'' contains dx_1.

Then it is easy to see that both $Q'\omega'$ and $Q\omega''$ are regular.

One computes

$$\begin{aligned} Q'(\omega'' \wedge dx_1) &= Q'(\omega \wedge dx_1) \\ &= (\omega \wedge dQ) - x_1(\omega \wedge dQ') \\ &= (-1)^q d(Q\omega) + (-1)^{q+1} Q(d\omega) - x_1(\omega \wedge dQ'). \end{aligned}$$

Since $\omega \in \Omega^q(\mathcal{A})$, $d(Q\omega)$ and $Q(d\omega)$ are both regular. Therefore $Q'(\omega'' \wedge dx_1)$ has no pole along $x_1 = 0$. Thus it is regular on V. Because ω'' contains no dx_1, $Q'\omega''$ is regular. ∎

Definition 2.2.2 For $\omega \in \Omega^q(\mathcal{A})$, the restriction of ω' in 2.2.1 to H_1 is called the **residue** of ω and is denoted by $\mathrm{res}(\omega)$.

The well-definedness of the residue of a logarithmic form is in [3, 2.4].

2.3 A short exact sequence

Let $\mathcal{A}$ be a generic arrangement described in 1.1.

Lemma 2.3.1 *For any $\omega \in \Omega^q(\mathcal{A})$, $\mathrm{res}(\omega) \in \Omega^{q-1}(\mathcal{A}'')$. In other words, one can define a $\mathbf{K}$-linear map*

$$\mathrm{res} : \Omega^q(\mathcal{A}) \to \Omega^{q-1}(\mathcal{A}'').$$

Proof. Write

$$\omega = \omega' \wedge (d\alpha_1/\alpha_1) + \omega''$$

as in 2.2.1. Since $d\omega \in \Omega^{q+1}(\mathcal{A})$, write

$$d\omega = \eta' \wedge (d\alpha_1/\alpha_1) + \eta'',$$

where both $Q'\eta'$ and $Q'\eta''$ are regular. Then one has

$$d\omega' \wedge (d\alpha_1/\alpha_1) + d\omega'' = d\omega = \eta' \wedge (d\alpha_1/\alpha_1) + \eta''.$$

Since both $Q'(d\omega')$ and $Q'(d\omega'')$ are regular,

$$\eta' \mid_{H_1} = \mathrm{res}(d\omega) = d\omega' \mid_{H_1} = d(\mathrm{res}(\omega))$$

because of the well-definedness of $\mathrm{res}(d\omega)$. Since $\mathcal{A}$ is generic, $d(\mathrm{res}(\omega))$ has at most a simple pole along $\mathcal{A}''$. Also it is obvious that $\mathrm{res}(\omega) = \omega' \mid_{H_1}$ has at most a simple pole along $\mathcal{A}''$. Therefore $\mathrm{res}(\omega) \in \Omega^{q-1}(\mathcal{A}'')$. ∎

Denote the inclusion map $\Omega^q(\mathcal{A}'') \hookrightarrow \Omega^q(\mathcal{A})$ by i.

Lemma 2.3.2 *The sequence*

$$0 \to \Omega^q(\mathcal{A}') \xrightarrow{i} \Omega^q(\mathcal{A}) \xrightarrow{\mathrm{res}} \Omega^{q-1}(\mathcal{A}'')$$

is exact for $q \geq 1$.

Proof. For $\omega \in \Omega^q(\mathcal{A}')$, we can choose $\omega' = 0$ and $\omega'' = \omega$ in 2.2.1. Thus $\mathrm{res}(\omega) = 0$.

We can assume that $\alpha_1 = x_1$. Write

$$\omega = \omega' \wedge (dx_1/x_1) + \omega''$$

as in the proof of 2.2.1; neither ω' nor ω'' contains dx_1. Suppose

$$\omega' \mid_{H_1} = \mathrm{res}(\omega) = 0.$$

Then every coefficient of ω' is divisible by x_1. So $Q'\omega$ is regular. Also $d\omega$ has no pole along H_1 either. Therefore $\omega \in \Omega^q(\mathcal{A}')$. ∎

Definition 2.3.3 An arrangement $\mathcal{A}$ in V is called **Boolean** if 1) $\#\mathcal{A} = n = \ell$, and 2) the intersection of all hyperplanes in $\mathcal{A}$ consists only of the origin. (In this case the intersection lattice $L(\mathcal{A})$ is a Boolean lattice.)

Theorem 2.3.4 *Let $0 \le q \le \ell - 2$. Let $\mathcal{A}$ be a Boolean or generic arrangement. Then $\Omega^q(\mathcal{A})$ is generated (as an S-algebra) by*

$$\{d\alpha/\alpha \mid \alpha \in V^*, \ker\alpha \in \mathcal{A}\}.$$

Proof. When $\mathcal{A}$ is Boolean it is easy to see (e. g., see [3, 2.9]) that the S-module $\Omega^1(\mathcal{A})$ is free with a basis $d\alpha_1/\alpha_1, \dots, d\alpha_\ell/\alpha_\ell$, where $\mathcal{A} = \{\ker\alpha_i \mid i = 1, \dots, \ell\}$. By [3, p. 270, THEOREM], we have

$$\Omega^q(\mathcal{A}) = \bigwedge^q \Omega^1(\mathcal{A}).$$

Suppose that $\mathcal{A}$ is generic. Define

$$F^q(\mathcal{A}) := \text{the submodule of } \Omega^q(\mathcal{A}) \text{ generated by } \{d\alpha/\alpha \mid \ker\alpha \in \mathcal{A}\}.$$

We shall show

$$\Omega^q(\mathcal{A}) = F^q(\mathcal{A})$$

by an induction on $n = \#\mathcal{A}$. Note that

$$\Omega^0(\mathcal{A}'') = F^0(\mathcal{A}'').$$

Since

$$\#\mathcal{A}' = \#\mathcal{A} - 1 \quad \text{and} \quad \#\mathcal{A}'' \le \#\mathcal{A} - 1,$$

we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \to & \Omega^q(\mathcal{A}') & \to & \Omega^q(\mathcal{A}) & \overset{\mathrm{res}}{\to} & \Omega^{q-1}(\mathcal{A}'') \\
 & & \| & & \cup\mid & & \| \\
0 & \to & F^q(\mathcal{A}') & \to & F^q(\mathcal{A}) & \overset{\mathrm{res}}{\to} & F^{q-1}(\mathcal{A}'')
\end{array}$$

by induction assumption. Let $\overline{S}$ be the algebra of all polynomial functions on H_1. Identify $\overline{S}$ with $S/\alpha_1 S$. For each $\overline{H} \in \mathcal{A}''$, there exists a unique $H \in \mathcal{A}'$ such that

$$H \cap H_1 = \overline{H}.$$

Let $\alpha_H \in V^*$ with $H = \ker \alpha_H$. Let $\overline{\alpha}_H$ be the residue class of α_H in $\overline{S}$. Then

$$\overline{H} = \ker(\overline{\alpha}_H).$$

Since

$$\mathrm{res}((d\alpha_H/\alpha_H) \wedge (d\alpha_1/\alpha_1)) = d\overline{\alpha}_H/\overline{\alpha}_H,$$

we have

$$\Omega^{q-1}(\mathcal{A}'') \supseteq \mathrm{res}\Omega^q(\mathcal{A}) \supseteq \mathrm{res}F^q(\mathcal{A}) = F^{q-1}(\mathcal{A}'') = \Omega^{q-1}(\mathcal{A}'').$$

Therefore the sequence

$$0 \to \Omega^q(\mathcal{A}') \xrightarrow{i} \Omega^q(\mathcal{A}) \xrightarrow{\mathrm{res}} \Omega^{q-1}(\mathcal{A}'') \to 0$$

is exact.

Let $\omega \in \Omega^q(\mathcal{A})$. Choose $\eta \in F^q(\mathcal{A})$ such that

$$\mathrm{res}(\omega) = \mathrm{res}(\eta).$$

Then

$$\omega - \eta \in \ker(\mathrm{res}) = \Omega^q(\mathcal{A}') = F^q(\mathcal{A}') \subseteq F^q(\mathcal{A}).$$

Therefore

$$\omega \in F^q(\mathcal{A}).$$

Thus

$$\Omega^q(\mathcal{A}) = F^q(\mathcal{A})$$

and the induction proceeds. ∎

Remark 2.3.5 Theorem 2.3.4 was proved by Ziegler in [7, 6.4.3] [8, 7.5] by a different method.

In the course of the proof of 2.3.4, we have already proved the following

Theorem 2.3.6 *Let $\mathcal{A}$ be generic and $1 \leq q \leq \ell - 2$. Then the sequence*

$$0 \to \Omega^q(\mathcal{A}') \xrightarrow{i} \Omega^q(\mathcal{A}) \xrightarrow{\mathrm{res}} \Omega^{q-1}(\mathcal{A}'') \to 0$$

is exact. ∎

3 $\Omega^q(\mathcal{A})$ $(0 \le q \le \ell - 2)$

3.1 The setup

We keep the setup of the previous section. Let $\mathcal{A} = \{H_1, H_2, \dots, H_n\}$ be a generic arrangement in V. Define

$$F_0 = F_0(\mathcal{A}) = \oplus_{i=1}^n Se_i$$

be a free S-module with a basis $e_1, e_2, \dots, e_n$. The grading of F_0 is introduced so that

$$\deg(e_i) = 0 \quad (1 \le i \le n).$$

Let $x_1, \dots, x_\ell$ be a basis for V^*. Choose $\alpha_i \in V^*$ with $\ker(\alpha_i) = H_i$. Let $\omega_i = d\alpha_i/\alpha_i$ for $1 \le i \le n$. Define an S-linear map

$$\varphi : F_0 \to \Omega^1(\mathcal{A})$$

by

$$\varphi(e_i) = \omega_i \qquad (1 \le i \le n).$$

Note that φ is homogeneous of degree zero. Define

$$F_1 = F_1(\mathcal{A}) = \ker \varphi.$$

3.2 The Fitting ideal of $\Omega^1(\mathcal{A})$

Restrict

$$\varphi : F_0 \to \Omega^1(\mathcal{A})$$

to a subspace $\oplus_{i=1}^n \mathbf{K}\alpha_i e_i$ to get

$$\psi : \oplus_{i=1}^n \mathbf{K}\alpha_i e_i \to T^*V := \oplus_{i=1}^\ell \mathbf{K} dx_i.$$

Then

$$\psi(\alpha_i e_i) = d\alpha_i \quad (1 \le i \le n).$$

Since $\mathcal{A}$ is generic, $d\alpha_1, \dots, d\alpha_n$ span T^*V. Denote $\ker(\psi)$ by B, which is a vector subspace of dimension $n - \ell$. Note that every element of B is homogeneous of degree one in F_0. Then the following theorem is due to Ziegler [7, 6.4.7] [8, 7.7]:

Theorem 3.2.1 *The sequence*

$$0 \to F_1 \to F_0 \xrightarrow{\varphi} \Omega^1(\mathcal{A}) \to 0$$

is a minimal free resolution of $\Omega^1(\mathcal{A})$. *Here* F_0 *is free of rank* n *and* F_1 *is free of rank* $n-\ell$. *In fact* F_1 *is naturally isomorphic to* $S \otimes_{\mathbf{K}} B$. ∎

The natural grading of F_1 is introduced so that F_1 may have a basis consisting of elements of degree zero. Then the inclusion map $F_1 \hookrightarrow F_0$ is homogeneous of degree $+1$. The Fitting ideal of $\Omega^1(\mathcal{A})$ is, by definition, generated by all $(n-\ell)$-minors of the matrix presenting the inclusion map $F_1 \hookrightarrow F_0$: All entries of the matrix are elemenets of degree one in S. Denote the Fitting ideal by $\mathcal{F}$.

Lemma 3.2.2 *Every product of* $n-\ell$ *elements in* $\{\alpha_1, \ldots, \alpha_n\}$ *lies in the Fitting ideal* $\mathcal{F}$.

Proof. We can assume that $\alpha_i = x_i \ \ (1 \le i \le \ell)$. Let

$$\alpha_i = c_{i1}x_1 + \cdots + c_{i\ell}x_\ell$$

with $c_{ij} \in \mathbf{K}$ for $\ell < i \le n, 1 \le j \le \ell$. Then

$$\beta_i = c_{i1}\alpha_1 e_1 + \cdots + c_{i\ell}\alpha_\ell e_\ell - \alpha_i e_i \quad (\ell < i \le n)$$

lies in $\ker(\varphi) = F_1$. Considering $\beta_{\ell+1}, \ldots, \beta_n$, one deduces that

$$\prod_{i=\ell+1}^{n} \alpha_i \in \mathcal{F}.$$

Since this is true for any numbering of $\alpha_1, \ldots, \alpha_n$, one gets the result. ∎

Theorem 3.2.3 *The Fitting ideal* $\mathcal{F}$ *is of height* ℓ.

Proof. If necessary, by extending the field $\mathbf{K}$, one can assume that $\mathbf{K}$ is algebraically closed. It is, by Hilbert's Nullstellensatz, enough to prove that the zero set $Z = Z(\mathcal{F}$ of the Fitting ideal $\mathcal{F}$ consists only of the origin. Let $v \in Z$. Suppose that

$$\alpha_i \begin{cases} = 0 & 1 \le i \le t, \\ \neq 0 & t < i \le n. \end{cases}$$

Then

$$\prod_{i=t+1}^{n} \alpha_i \notin \mathcal{F}.$$

By 3.2.2, one has $n - t < n - \ell$. Thus $t > \ell$. So

$$v \in \bigcap_{i=1}^{t} H_i = \{0\}. \blacksquare$$

3.3 $\bigwedge^q \Omega^1(\mathcal{A}) \simeq \Omega^q(\mathcal{A})$

By applying the result of Lebelt [1, p. 191, Folgerung 1 a), b)], from 3.2.3 we obtain

Theorem 3.3.1 $\bigwedge^q \Omega^1(\mathcal{A})$ *is torsion-free for* $0 \leq q \leq \ell - 1$. $\blacksquare$

Theorem 3.3.2 *For* $0 \leq q \leq \ell - 2$, *there exists a natural isomorphism*

$$p : \bigwedge^q \Omega^1(\mathcal{A}) \xrightarrow{\sim} \Omega^q(\mathcal{A}).$$

Proof. By 2.3.4 the natural map

$$p : \bigwedge^q \Omega^1(\mathcal{A}) \to \Omega^q(\mathcal{A})$$

is surjective. Let ξ be the map of multiplying Q^q, where $Q = \prod_{i=1}^{n} \alpha_i$. Then we have a commutative diagram

$$\begin{array}{ccc} \bigwedge^q \Omega^1(\mathcal{A}) & \xrightarrow{p} & \Omega^q(\mathcal{A}) \\ \downarrow \xi & & \downarrow \xi \\ \bigwedge^q \Omega^1_S & \xrightarrow{p'} & \Omega^q_S \end{array}$$

because $Q\Omega^1(\mathcal{A}) \subseteq \Omega^1_S$. Here p' is also the natural map. Since ξ is injective by 3.3.1 and p' is isomorphic, one knows that p is also injective. $\blacksquare$

3.4 Free resolution of $\Omega^q(\mathcal{A})$ $\quad (0 \leq q \leq \ell - 2)$

In [1, p.191, Folgerung 1 b)] and [2, p. 345, Beispiele (ii)], Lebelt constructed a free resolution of the exterior power of a module of homological dimension one. Let S^p denote the p-th symmetric power. (Note that the symmetric power is isomorphic to the divided power in [2, p. 343] because the field $\mathbf{K}$ is of characteristic zero.) Since we have 3.2.3 and 3.3.2, we have a free resolution $(C^{\cdot}(\Omega^q(\mathcal{A})), d)$ of $\Omega^q(\mathcal{A})$ for $0 \leq q \leq \ell - 2$:

$$C^p(\Omega^q(\mathcal{A})) = S^p F_1 \otimes \bigwedge^{q-p} F_0$$

for $0 \leq p \leq q$. Here the tensor product is over S. The boundary map

$$d : S^p F_1 \otimes \bigwedge^{q-p} F_0 \to S^{p-1} F_1 \otimes \bigwedge^{q-p+1} F_0$$

is defined by the S-linear map satisfying

$$d(y_1 \cdots y_p \otimes \beta) = \sum_{i=1}^{p} y_1 \cdots \hat{y}_i \cdots y_p \otimes (y_i \wedge \beta),$$

where $y_i \in F_1$ and $\beta \in \bigwedge^{q-p} F_0$ for $0 \leq p \leq q$. This map d is homogeneous of degree $+1$. Also we define the augmented map

$$\varepsilon : C^0(\Omega^q(\mathcal{A})) = \bigwedge^{q} F_0 \to \Omega^q(\mathcal{A})$$

by the S-linear map satisfying

$$\varepsilon(e_{i_1} \wedge \cdots \wedge e_{i_q}) = \omega_{i_1} \wedge \cdots \wedge \omega_{i_q}$$

for $1 \leq i_1 < \ldots < i_q \leq n$. Then this map ε is homogeneous of degree zero. Thanks to 3.3.2 we have

Theorem 3.4.1 *The complex $(C^{\cdot}(\Omega^q(\mathcal{A})), d)$ is a minimal resolution of $\Omega^q(\mathcal{A})$ for $0 \leq q \leq \ell - 2$.* ∎

Since $F_0 \simeq S^n$ and $F_1 \simeq S^{n-\ell}$ (3.2.1), the rank of a free module $C^p(\Omega^q(\mathcal{A}))$ is equal to $\binom{n-\ell+p-1}{p}\binom{n}{q-p}$. Let i and d be integers. Define the graded S-module $S(i)$ by:

$$S(i)_d = S_{i+d}.$$

Then we have

Corollary 3.4.2 *Let $0 \leq q \leq \ell - 2$. There exists an exact sequence*

$$0 \to S(-q)^{v(q)} \to \ldots \to S(-p)^{v(p)} \to \ldots \to S(-1)^{v(1)} \to S^{v(0)} \to \Omega^q(\mathcal{A}) \to 0,$$

where

$$v(p) = \binom{n-\ell+p-1}{p}\binom{n}{q-p} \quad (0 \leq p \leq q)$$

and all maps are homogeneous of degree zero. ∎

Corollary 3.4.3 *The homological dimension of $\Omega^q(\mathcal{A})$ is equal to q for $0 \leq q \leq \ell - 2$.* ∎

4 $\Omega^{\ell-1}(\mathcal{A})$

4.1 The setup

We keep the setup of the previous section. Let

$$F_0' = F_0'(\mathcal{A}) = \oplus_{i=2}^{n} S e_i.$$

Then F_0' is a free S-module of rank $n-1$.

4.2 Exact sequence $(\Omega^{\cdot}(\mathcal{A}), \partial)$

Definition 4.2.1 Let $\theta_E : S \to S$ be the **Euler derivation**:

$$\theta_E = \sum_{i=1}^{\ell} x_i (\partial / \partial x_i).$$

For $\omega \in \Omega^q(\mathcal{A})$, define a $(q-1)$-form $< \omega, \theta_E >$ such that
1. the correspondence $\omega \mapsto < \omega, \theta_E >$ is S-linear,
2.

$$< dx_{i_1} \wedge \cdots \wedge dx_{i_q}, \theta_E > = \sum_{p=1}^{q} (-1)^{p+1} x_{i_p} dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_p}} \wedge \cdots \wedge dx_{i_q}.$$

This definition is independent of the choice of a basis $x_1, \ldots, x_\ell$ for V^*.

Lemma 4.2.2 *For any $\omega \in \Omega^q(\mathcal{A})$ and $\alpha \in V^*$,*

$$< d\alpha \wedge \omega, \theta_E >= \alpha\omega - d\alpha \wedge < \omega, \theta_E > .$$

Proof. We can assume that

$$\alpha = x_1 \quad \text{and} \quad \omega = dx_{i_1} \wedge \cdots \wedge dx_{i_q}$$

for $1 \leq i_1 < i_2 < \cdots < i_q \leq \ell$.

case 1. $(1 < i_1)$

$$\begin{aligned} LHS &= < dx_1 \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}, \theta_E > \\ &= x_1 dx_{i_1} \wedge \cdots \wedge dx_{i_q} - dx_1 \wedge < dx_{i_1} \wedge \cdots \wedge dx_{i_q}, \theta_E >= RHS. \end{aligned}$$

case 2. $(1 = i_1)$

$$RHS = x_1 dx_{i_1} \wedge \cdots \wedge dx_{i_q} - dx_1 \wedge (x_1 dx_{i_2} \wedge \cdots \wedge dx_{i_q}) = 0 = LHS. \blacksquare$$

Lemma 4.2.3 *For any $\omega \in \Omega^q(\mathcal{A})$, $< \omega, \theta_E > \in \Omega^{q-1}(\mathcal{A})$.*

Proof. Let $\alpha \in V^*$ with $H = \ker \alpha \in \mathcal{A}$. By 4.2.2, we know

$$d\alpha \wedge < \omega, \theta_E >= \alpha\omega - < d\alpha \wedge \omega, \theta_E > .$$

This has no pole along H. So $< \omega, \theta_E > \in \Omega^{q-1}(\mathcal{A})$. $\blacksquare$

Let $\omega_1 = d\alpha_1/\alpha_1$. Define

$$\partial : \Omega^q(\mathcal{A}) \to \Omega^{q+1}(\mathcal{A})$$

by

$$\partial(\omega) = \omega_1 \wedge \omega$$

for $\omega \in \Omega^q(\mathcal{A})$.

Theorem 4.2.4 *The sequnce*

$$0 \to S \xrightarrow{\partial} \Omega^1(\mathcal{A}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^\ell(\mathcal{A}) \to 0.$$

is exact.

Proof. If $\omega \in \Omega^q(\mathcal{A})$ satisfies $\omega_1 \wedge \omega = \partial\omega = 0$, then we have

$$\partial < \omega, \theta_E >= \omega_1 \wedge < \omega, \theta_E >= \omega - < \omega_1 \wedge \omega, \theta_E >= \omega$$

by 4.2.2. Since $< \omega, \theta_E >$ lies in $\Omega^{q-1}(\mathcal{A})$ by 4.2.3, the sequence is exact. $\blacksquare$

4.3 Free resolution of $\Omega^{\ell-1}(\mathcal{A})$

Recall the free module $F_0' = \oplus_{i=2}^n Se_i$. Let $0 \le q \le \ell - 2$. Then

$$F_0 = F_0' \oplus Se_1$$

and

$$\bigwedge^q F_0 = (\bigwedge^q F_0') \oplus (e_1 \wedge \bigwedge^{q-1} F_0').$$

Define

$$\partial_q : \bigwedge^q F_0 \to \bigwedge^q F_0'$$

by the first projection:

$$\partial_q(x + (e_1 \wedge y)) = x,$$

where $x \in \bigwedge^q F_0'$ and $y \in \bigwedge^{q-1} F_0'$.

Define

$$\partial_k : \bigwedge^k F_0 \to \bigwedge^{k+1} F_0 \quad (0 \le k < q)$$

by $\partial_k(x) = e_1 \wedge x$. Then the following lemma is easy:

Lemma 4.3.1 *The sequence*

$$0 \to S \xrightarrow{\partial_0} F_0 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{q-1}} \bigwedge^q F_0 \xrightarrow{\partial_q} \bigwedge^q F_0' \to 0$$

is exact. ∎

Let $p = \ell - 2 - q$. By tensoring (over S) the $(-1)^p$-multiplication of the p-th symmetric power S^pF_1 to ∂_k , we obtain

$$(-1)^p \otimes \partial_k : S^pF_1 \otimes \bigwedge^k F_0 \to S^pF_1 \otimes \bigwedge^{k+1} F_0 \quad (0 \le k < q).$$

Also by tensoring the identity map of S^pF_1 to ∂_q , we obtain

$$1 \otimes \partial_q : S^pF_1 \otimes \bigwedge^q F_0 \to S^pF_1 \otimes \bigwedge^q F_0'.$$

For simplicity denote all these maps by ∂:

$$\partial = (-1)^p \otimes \partial_k \quad (0 \le k < q), \quad \partial = 1 \otimes \partial_q.$$

Then we have

Theorem 4.3.2 *Let $0 \le p \le \ell - 2$ and $q = \ell - 2 - p$. Then the sequence*

$$0 \to S^p F_1 \xrightarrow{\partial} S^p F_1 \otimes F_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} S^p F_1 \otimes \bigwedge^q F_0 \xrightarrow{\partial} S^p F_1 \otimes \bigwedge^q F_0' \to 0$$

is exact. ∎

For simplicity we write

$$\begin{aligned} C^{p,q} &= C^p(\Omega^q(\mathcal{A})) = S^p F_1 \otimes \bigwedge^{q-p} F_0 \quad (0 \le p \le q \le \ell - 2), \\ D^p &= S^p F_1 \otimes \bigwedge^{\ell-2-p} F_0' \quad (0 \le p \le \ell - 2). \end{aligned}$$

Then by 4.3.2 the sequence

$$0 \to C^{p,p} \xrightarrow{\partial} C^{p,p+1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C^{p,\ell-2} \xrightarrow{\partial} D^p \to 0$$

is exact for $0 \le p \le \ell - 2$. Integrating these exact sequences with those in 3.4.1 and 4.2.4 we obtain a big diagram

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \uparrow & & & & \\
0 & \to & S & \xrightarrow{\partial} & \Omega^1(\mathcal{A}) & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \Omega^{\ell-2}(\mathcal{A}) & \xrightarrow{\partial} & \omega_1 \wedge \Omega^{\ell-2}(\mathcal{A}) & \to & 0 \\
 & & \varepsilon \uparrow & & \varepsilon \uparrow & & & & \varepsilon \uparrow & & & & \\
0 & \to & C^{0,0} & \xrightarrow{\partial} & C^{0,1} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & C^{0,\ell-2} & \xrightarrow{\partial} & D^0 & \to & 0 \\
 & & \uparrow & & d \uparrow & & & & d \uparrow & & & & \\
 & & 0 & \to & C^{1,1} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & C^{1,\ell-2} & \xrightarrow{\partial} & D^1 & \to & 0 \\
 & & & & \uparrow & & & & d \uparrow & & & & \\
 & & & & 0 & & & & \cdot & & \cdot & & \cdot \\
 & & & & & & & & \cdot & & \cdot & & \cdot \\
 & & & & & & & & d \uparrow & & & & \\
 & & & & & & 0 & \to & C^{\ell-2,\ell-2} & \xrightarrow{\partial} & D^{\ell-2} & \to & 0 \\
 & & & & & & & & \uparrow & & & & \\
 & & & & & & & & 0 & & & & .
\end{array}$$

Here all the columns and the rows are exact.

Lemma 4.3.3 *The big diagram above is commutative.*

Proof. Let

$$y_1 \cdots y_p \otimes z \in C^{p,q} = S^p F_1 \otimes \bigwedge^{q-p} F_0$$

for $1 \le p \le q \le \ell - 2$. Then

$$\begin{aligned}
d \circ \partial(y_1 \cdots y_p \otimes z) &= d((-1)^p y_1 \cdots y_p \otimes (e_1 \wedge z)) \\
&= \sum_{i=1}^{p} (-1)^p y_1 \cdots \widehat{y_i} \cdots y_p \otimes (y_i \wedge e_1 \wedge z) \\
&= \sum_{i=1}^{p} (-1)^{p-1} y_1 \cdots \widehat{y_i} \cdots y_p \otimes (e_1 \wedge y_i \wedge z) \\
&= \partial \left(\sum_{i=1}^{p} y_1 \cdots \widehat{y_i} \cdots y_p \otimes (y_i \wedge z) \right) \\
&= \partial \circ d(y_1 \cdots y_p \otimes z).
\end{aligned}$$

Next let

$$z \in C^{0,q} = \bigwedge^{q} F_0$$

for $0 \le q \le \ell - 2$. Then

$$\partial \circ \varepsilon(z) = \omega_1 \wedge \varepsilon(z) = \varepsilon(e_1 \wedge z) = \varepsilon \circ \partial(z). \blacksquare$$

We shall fill the column of the D^p's of the big diagram with S-linear maps:

$$d : D^p \to D^{p-1} \quad (1 \le p \le \ell - 2)$$

satisfying

$$d(y_1 \cdots y_p \otimes z) = \sum_{i=1}^{p} y_1 \cdots \widehat{y_i} \cdots y_p \otimes [\pi(y_i) \wedge z]$$

for $y_i \in F_1$, $z \in \bigwedge^{\ell-2-p} F_0'$. Here

$$\pi : F_1 \to F_0'$$

is the restriction of the first projection map

$$F_0 = F_0' \oplus S e_1 \to F_0'$$

to F_1. Then all of these maps d are homogeneous of degree $+1$. Also let

$$\varepsilon : D^0 = \bigwedge^{\ell-2} F_0' \to \omega_1 \wedge \Omega^{\ell-2}(\mathcal{A})$$

be the S-linear map satisfying

$$\varepsilon(e_{i_1} \wedge \cdots \wedge e_{i_{\ell-2}}) = \omega_1 \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{i_{\ell-2}},$$

where $1 < i_1 < \ldots < i_{\ell-2} \leq n$. Then this map ε is homogeneous of degree zero. Then we have

Lemma 4.3.4 *The sequence*

$$0 \to D^{\ell-2} \xrightarrow{d} D^{\ell-1} \xrightarrow{d} \cdots \xrightarrow{d} D^0 \xrightarrow{\varepsilon} \omega_1 \wedge \Omega^{\ell-2}(\mathcal{A}) \to 0$$

gives a minimal resolution of the S-module $\omega_1 \wedge \Omega^{\ell-2}(\mathcal{A})$.

Proof. Since all of these maps d are homogeneous of degree $+1$, the minimality of the resolution is obtained by applying a well-known minimality criterion (e. g., [5, Lemma 4.4]). So it is enough to show that the insertion of the maps d and ε to the big diagram keeps its commutativity. First we shall prove the diagram

$$\begin{array}{ccc} C^{p-1,\ell-2} & \xrightarrow{\partial} & D^{p-1} \\ d\uparrow & & \uparrow d \\ C^{p,\ell-2} & \xrightarrow{\partial} & D^{p} \end{array}$$

is commutative. Let

$$y_1 \cdots y_p \otimes z \in C^{p,\ell-2} = S^p F_1 \otimes \bigwedge^{\ell-2-p} F_0$$

for $y_i \in F_1$ and $z \in \bigwedge^{\ell-2-p} F_0$. Write

$$z = z_1 + (e_1 \wedge z_2)$$

with $z_1 \in \bigwedge^{\ell-2-p} F_0'$ and $z_2 \in \bigwedge^{\ell-3-p} F_0'$. Then

$$\begin{aligned} \partial \circ d(y_1 \cdots y_p \otimes z) &= \partial\left(\sum_{i=1}^{p} y_1 \cdots \widehat{y_i} \cdots y_p \otimes ((y_i \wedge z_1) + (y_i \wedge e_1 \wedge z_2))\right) \\ &= \sum_{i=1}^{p} y_1 \cdots \widehat{y_i} \cdots y_p \otimes (y_i \wedge z_1) \\ &= d(y_1 \cdots y_p \otimes z_1) \\ &= d \circ \partial(y_1 \cdots y_p \otimes (z_1 + (e_1 \wedge z_2))) \\ &= d \circ \partial(y_1 \cdots y_p \otimes z). \end{aligned}$$

Next we shall prove that the diagram

$$\begin{array}{ccc} \Omega^{\ell-2}(\mathcal{A}) & \xrightarrow{\partial} & \omega_1 \wedge \Omega^{\ell-2}(\mathcal{A}) \\ \varepsilon \uparrow & & \uparrow \varepsilon \\ C^{0,\ell-2} & \xrightarrow{\partial} & D^0 \end{array}$$

is commutative. Let

$$e_{i_1} \wedge \cdots \wedge e_{i_{\ell-2}} \in C^{0,\ell-2} = \bigwedge^{\ell-2} F_0,$$

for $1 \leq i_1 < \ldots < i_{\ell-2} \leq n$. Then

$$\begin{aligned} \partial \circ \varepsilon(e_{i_1} \wedge \cdots \wedge e_{i_{\ell-2}}) &= \partial(\omega_{i_1} \wedge \cdots \wedge \omega_{i_{\ell-2}}) \\ &= \omega_1 \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{i_{\ell-2}} \\ &= \varepsilon \circ \partial(e_{i_1} \wedge \cdots \wedge e_{i_{\ell-2}}). \blacksquare \end{aligned}$$

Definition 4.3.5 Define

$$\omega_E = \left(\sum_{i=1}^{\ell} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_\ell \right) / Q.$$

Then $\omega_E \in \Omega^{\ell-1}(\mathcal{A})$.

Note that ω_E is independent of the choice of a basis $x_1, \ldots, x_\ell$ and that the degree of ω_E is equal to $\ell - n$.

Lemma 4.3.6

$$\Omega^{\ell-1}(\mathcal{A}) = (\omega_1 \wedge \Omega^{\ell-2}(\mathcal{A})) \oplus S\omega_E.$$

Proof. Recall the sequence 4.2.4

$$0 \to \Omega^0(\mathcal{A}) \xrightarrow{\partial} \Omega^1(\mathcal{A}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^\ell(\mathcal{A}) \to 0.$$

Since

$$\partial(\omega_E) = \omega_1 \wedge \omega_E = (dx_1 \wedge \cdots \wedge dx_\ell)/Q$$

and

$$\Omega^\ell(\mathcal{A}) = S(dx_1 \wedge \cdots \wedge dx_\ell)/Q \quad \text{(free of rank one)},$$

we have a short exact sequence

$$0 \to \partial\Omega^{\ell-2}(\mathcal{A}) \to \Omega^{\ell-1}(\mathcal{A}) \xrightarrow{\partial} \Omega^{\ell}(\mathcal{A}) \to 0$$

which splits. Thus we have the desired result. ■

Define $\varepsilon' : S \to \Omega^{\ell-1}(\mathcal{A})$ by $\varepsilon'(f) = f\omega_E$ for $f \in S$. Then ε' is homogeneous of degree $\ell - n$. Combining the free resolution 4.3.4 of $\omega_1 \wedge \Omega^{\ell-2}(\mathcal{A})$ and 4.3.6, we finally have

Theorem 4.3.7 *The sequence*

$$0 \to D^{\ell-2} \xrightarrow{d} D^{\ell-1} \xrightarrow{d} \cdots \xrightarrow{d} D^1 \xrightarrow{(d,0)} D^0 \oplus S \xrightarrow{(\varepsilon,\varepsilon')} \Omega^{\ell-1}(\mathcal{A}) \to 0$$

gives a minimal free resolution of the S-module $\Omega^{\ell-1}(\mathcal{A})$. ■

Since $F_0' \simeq S^{n-1}$ and $F_1 \simeq S^{n-\ell}$, the rank of a free module $D^p = S^pF_1 \otimes \bigwedge^{\ell-2-p} F_0'$ is equal to $\binom{n-\ell+p-1}{p}\binom{n-1}{\ell-p-2}$. Considering the degrees of the maps d and d', we have

Corollary 4.3.8 *There exists an exact sequence*

$$\begin{aligned}&0 \to S(2-\ell)^{w(\ell-2)} \to \cdots \to S(-p)^{w(p)} \to \cdots \to S(-1)^{w(1)} \\ &\to S^{w(0)} \oplus S(n-\ell) \to \Omega^{\ell-1}(\mathcal{A}) \to 0,\end{aligned}$$

where $w(p) = \binom{n-\ell+p-1}{p}\binom{n-1}{\ell-p-2}$ $(0 \le p \le \ell-2)$ *and all the maps are homogeneous of degree zero.* ■

Corollary 4.3.9 *The homological dimension of the S-module $\Omega^{\ell-1}(\mathcal{A})$ is equal to $\ell-2$.* ■

4.4 Remark on $D(\mathcal{A})$

Let $\mathrm{Der} = \mathrm{Der}_{\mathbf{K}}(S)$ denote the module of all $\mathbf{K}$-derivations on S:

$$\begin{aligned}\mathrm{Der} = \{\theta \mid\ & \theta : S \to S \text{ is } \mathbf{K}\text{-linear and} \\ & \theta(fg) = f\theta(g) + g\theta(f) \text{ for all } f, g \in S\}.\end{aligned}$$

An element $\theta \in \mathrm{Der}$ is called to be homogeneous of degree d if $\theta(x) \in S_{d+1}$ for all $x \in V^*$. Then it is easy to see that Der is naturally a graded free S-module of rank ℓ.

Define

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der} \mid \theta(Q) \in QS\}$$

for $Q = Q(\mathcal{A})$. Then $D(\mathcal{A})$ is a graded submodule of Der. The S-module $D(\mathcal{A})$ is called the module of logarithmic derivations along $\mathcal{A}$. It has been studied in [4] and others. Recall the Euler derivation

$$\theta_E = \sum_{i=1}^{\ell} x_i(\partial/\partial x_i)$$

in 4.2.1 and

$$\omega_E = \left(\sum_{i=1}^{\ell}(-1)^{i-1}x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_\ell\right)/Q$$

in 4.3.5. Then $\deg\theta_E = 0$ and $\deg\omega_E = \ell - n$.

The following lemma is easy:

Lemma 4.4.1 *There is an isomorphism*

$$\gamma : D(\mathcal{A}) \xrightarrow{\sim} \Omega^{\ell-1}(\mathcal{A})$$

defined by

$$\gamma(\theta) = \left(\sum_{i=1}^{\ell}(-1)^{i-1}\theta(x_i) dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_\ell\right)/Q$$

for $\theta \in D(\mathcal{A})$. Also $\gamma(\theta_E) = \omega_E$. ∎

By 4.3.7, we have:

Theorem 4.4.2 *There exists a minimal free resolution*

$$0 \to D^{\ell-2} \xrightarrow{d} D^{\ell-1} \xrightarrow{d} \cdots \xrightarrow{d} D^1 \xrightarrow{(d,0)} D^0 \oplus S \to D(\mathcal{A}) \to 0$$

of the S-module $D(\mathcal{A})$. ∎

Since the map γ is homogeneous of degree $\ell - n$, we also have

Corollary 4.4.3 *There exists an exact sequence*

$$
\begin{aligned}
&0 \to S(2-n)^{w(\ell-2)} \to \cdots \to S(\ell-n-p)^{w(p)} \to \cdots \to S(\ell-n-1)^{w(1)} \\
&\to S(\ell-n)^{w(0)} \oplus S \to D(\mathcal{A}) \to 0,
\end{aligned}
$$

where $w(p) = \binom{n-\ell+p-1}{p}\binom{n-1}{\ell-p-2}$ $(0 \le p \le \ell-2)$ *and all the maps are homogeneous of degree zero.* ∎

Corollary 4.4.4 *The characteristic sequence (see [6, p. 315]) of a generic arrangement (with n hyperplanes in an ℓ-dimensional vector space) is:*

$$
(1, \underbrace{n-\ell+1, \ldots, n-\ell+1}_{\binom{n-1}{\ell-2}}; \ldots; \underbrace{n-\ell+k, \ldots, n-\ell+k}_{\binom{n-\ell+k-2}{k-1}\binom{n-1}{\ell-1-k}}; \ldots
$$
$$
; \underbrace{n-1, \ldots, n-1}_{\binom{n-3}{\ell-2}}). \blacksquare
$$

Corollary 4.4.5 *The homological dimension of the S-module $D(\mathcal{A})$ is equal to $\ell - 2$.* ∎

4.5 Remark on the factor ring S/J

Consider

$$\mathrm{Ann}(\mathcal{A}) = \{\theta \in D(\mathcal{A}) \mid \theta(Q) = 0\}.$$

Then $\mathrm{Ann}(\mathcal{A})$ is a graded submodule of $D(\mathcal{A})$. Let $J = J(Q)$ be the Jacobian ideal of Q:

$$J = (\partial Q/\partial x_1, \ldots, \partial Q/\partial x_\ell)S.$$

Define an S-linear map

$$\mu : \mathrm{Der} \to S$$

by $\mu(\theta) = \theta(Q)$ for $\theta \in \mathrm{Der}$. Then it is easy to see

Lemma 4.5.1 *The sequence*

$$0 \to \mathrm{Ann}(\mathcal{A}) \to \mathrm{Der} \xrightarrow{\mu} S \to S/J \to 0$$

is exact. ∎

Lemma 4.5.2 *(1)* $D(\mathcal{A}) = \mathrm{Ann}(\mathcal{A}) \oplus S\theta_E$*, and (2) there exists an* S*-linear isomorphism*

$$\tau : \ \omega_1 \wedge \Omega^{\ell-2}(\mathcal{A}) \stackrel{\sim}{\to} \mathrm{Ann}(\mathcal{A}).$$

Proof.

(1) Let $\theta \in D(\mathcal{A})$. Note that $n = \deg Q$. Then

$$\theta - (\theta(Q)/nQ)\theta_E \in \mathrm{Ann}(\mathcal{A}).$$

If $f\theta_E \in \mathrm{Ann}(\mathcal{A}) \cap S\theta_E$, then

$$0 = f\theta_E(Q) = nfQ.$$

Thus $f = 0$.

(2) Recall the isomorphism

$$\gamma : D(\mathcal{A}) \stackrel{\sim}{\to} \Omega^{\ell-1}(\mathcal{A})$$

in 4.4.1. Since $\gamma(\theta_E) = \omega_E$, one has

$$\Omega^{\ell-1}(\mathcal{A}) = \gamma(\mathrm{Ann}(\mathcal{A})) \oplus S\omega_E$$

by (1). Recall 4.3.6:

$$\Omega^{\ell-1}(\mathcal{A}) = (\omega_1 \wedge \Omega^{\ell-2}(\mathcal{A})) \oplus S\omega_E.$$

Thus we have (2). ∎

From 4.3.4, 4.5.2 (2), and 4.5.1, one has

Theorem 4.5.3 *There exists a minimal free resolution*

$$0 \to D^{\ell-2} \xrightarrow{d} D^{\ell-1} \xrightarrow{d} \cdots \xrightarrow{d} D^0 \to \mathrm{Der} \xrightarrow{\mu} S \to S/J \to 0$$

of the S*-module* S/J. ∎

Considering the degrees of the maps, one obtains

Corollary 4.5.4 *There exists an exact sequence*

$$0 \to S(2-2n)^{w(\ell-2)} \to \cdots \to S(\ell-2n-p)^{w(p)} \to \cdots \to S(\ell-2n)^{w(0)}$$
$$\to S(1-n)^{\ell} \to S \to S/J \to 0,$$

where $w(p) = \begin{pmatrix} n-\ell+p-1 \\ p \end{pmatrix} \begin{pmatrix} n-1 \\ \ell-p-2 \end{pmatrix}$ $(0 \le p \le \ell-2)$ *and all the maps are homogeneous of degree zero.* ∎

Corollary 4.5.5 *The homological dimension of the* S*-module* S/J *is equal to* ℓ*. Therefore the depth of* S/J *is equal to zero.* ∎

Example 4.5.6 Let $Q = (x+y+z)xyz$. Then Q defines a generic arrangement with $\ell = 3$ and $n = 4$. Then the Jacobian ideal J is defined by

$$J = (yz(2x+y+z), zx(x+2y+z), xy(x+y+2z))S.$$

Let $\alpha_1 = x+y+z$, $\alpha_2 = x$, $\alpha_3 = y$, $\alpha_4 = z$, and $\omega_i = d\alpha_i/\alpha_i$ $(i = 1,2,3,4)$. Then

$$\begin{aligned}
F_0 &= Se_1 \oplus Se_2 \oplus Se_3 \oplus Se_4, \\
F_0' &= Se_2 \oplus Se_3 \oplus Se_4, \\
F_1 &= S(xe_2 + ye_3 + ze_4 - (x+y+z)e_1), \\
D^0 &= F_0', \\
D^1 &= F_1.
\end{aligned}$$

As in 4.3.4, a minimal free resolution of the S-module $\omega_1 \wedge \Omega^1(\mathcal{A})$ is given by:

$$0 \to F_1 \xrightarrow{d} F_0' \xrightarrow{\varepsilon} \omega_1 \wedge \Omega^1(\mathcal{A}) \to 0.$$

Here

$$d(xe_2 + ye_3 + ze_4 - (x+y+z)e_1) = xe_2 + ye_3 + ze_4,$$

$$\begin{aligned}
\varepsilon(e_2) &= (d(x+y+z)/(x+y+z)) \wedge (dx/x), \\
\varepsilon(e_3) &= (d(x+y+z)/(x+y+z)) \wedge (dy/y), \\
\varepsilon(e_4) &= (d(x+y+z)/(x+y+z)) \wedge (dz/z).
\end{aligned}$$

The isomorphism (4.5.2 (2))

$$\tau : \omega_1 \wedge \Omega^{\ell-2}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Ann}(\mathcal{A})$$

in this case satisfies

$$\begin{aligned}
\tau \circ \varepsilon(e_2) &= \omega_1 \wedge \omega_2 = yz((\partial/\partial y) - (\partial/\partial z)) + \frac{1}{4}(y - z)\theta_E,\\
\tau \circ \varepsilon(e_3) &= \omega_1 \wedge \omega_3 = zx((\partial/\partial z) - (\partial/\partial x)) + \frac{1}{4}(z - x)\theta_E,\\
\tau \circ \varepsilon(e_4) &= \omega_1 \wedge \omega_4 = xy((\partial/\partial x) - (\partial/\partial y)) + \frac{1}{4}(x - y)\theta_E.
\end{aligned}$$

The map

$$\mu : \mathrm{Der} \to S$$

is defined by $\mu(\theta) = \theta(Q)$. Then the minimal free resolution of S/J in 4.5.3 is:

$$0 \to F_1 \xrightarrow{d} F_0' \xrightarrow{\tau\circ\varepsilon} \mathrm{Der} \xrightarrow{\mu} S \to S/J \to 0.$$

So we get an exact sequence

$$0 \to S(-6) \to S(-5)^3 \to S(-3)^3 \to S \to S/J \to 0,$$

where all the maps are homogeneous of degree zero. This is the exact sequence in Corollary 4.5.4.

4.6 Remark on positive characteristic

When $\mathbb{K}$ has a positive characteristic, the symmetric power so far should be replaced by the divided power as in [2]. Then all the results up to 4.4 hold true after a suitable change of the definition of the boundary map d [2, p. 345, Beispiele (ii)]. Unless n is a multiple of the characteristic, the results in 4.5 are still true. (When n is a multiple of the characteristic, 4.5.2 is not true: $\theta_E \in \mathrm{Ann}(\mathcal{A})$.)

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Deformation theory of isolated singularities from the point of view of CR-structures (the deformation theory of pseudo hermitian CR-structures which preserve the Webster's scalar curvature)

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Abstract Recently , for compact projective manifolds , as a generalization of the Teichmüler space to the higher dimensional case , the global moduli theory has been developed by several authors (Siu , Fujiki and Schumacher (Si) , (Sch)) . On the other hand , for open manifolds , nothing has been known . And last thirty years , several similarities between project algebraic spaces and strongly pseudo convex spaces have been shown . Therefore it seems natural to try to construct a global moduli theory for strongly pseudo convex spaces .

Let X be a strongly pseudo convex space and let r be a C^{∞} strictly pluri-subharmonic function except a compact subset . Let

$$\Omega = \{ x : x \in X , r(x) < 0 \}$$

and let $b\Omega$ be its boundary . Then over $b\Omega$, a CR-structure is induced from X . Namely let

$$^{o}T'' = C\otimes T(b\Omega) \cap T''X|_{b\Omega} , \quad M = b\Omega .$$

Then the pair $(M, {}^{o}T'')$ is called a CR-structure . I should expain why we consider such an abstract object . Because , our Ω might have singularities . And furthermore Ω is open (not compact) , so these facts cause several troubles in using analysis . However very fortunately , the CR-structure $(M, {}^{o}T'')$ determines Ω , almostly (for example see Rossi's theorem) . And furthermore $(M, {}^{o}T'')$ and Ω with the boundary $b\Omega$ have the simlar property in analysis (if Ω has no singularity) . And technically

$(M, {}^{o}T'')$ can be handeled much easier than Ω . Of course CR-structure itself is interesting . But from our stand point , we are always considering Ω in mind .

As you know , for a strongly pseudo convex space , as for local theory , for the first time , a versal family in a sense of Kuranishi is constructed by (Ak1),(Ak2) under several assumptions from the point of view of CR-structures . Later , by a complete different method , without any assumptions , by (Bi-Ko) , the existence of the versal family is shown . Therefore nowadays , it is not necessary to use CR-structures in the local moduli theory . However , in the global moduli theory for a compact case , a real analysis method is essentially used and indispensable . Therefore we might hope that the CR-structure method could revive in the global theory .

The first difficulty for constructing a global moduli space is that : the parameter space of the local versal family may not be the local moduli space . Of course this phenomenon appears in the compact complex manifold case . But for a compact complex ,anifold X , if we assume :

$$H^{0}(X, \Theta_{X}) = 0 \quad ,$$

where Θ_X means the holomorphic tangent bundle , then the local versal family must be the moduli family . Contrast to the compact complex manifold case , we can't expect such a theorem in the CR-case . Namely , our stand point is : we are always considering open complex spaces . In this sense , if we are given a family of CR-structures $(M, {}^{\phi(t)}T'')$, $t \in T$, and t_1 , $t_2 \in T$, satisfying

$$(V_{t_1}, A_{t_1}) \xrightarrow{\sim} (V_{t_2}, A_{t_2}) \quad \text{as a germ of singularities } A_{t_1} \text{ and } A_{t_2} \; ;$$

where (V_{t_1}, A_{t_1}) is a normal stein space determined by $(M, {}^{\phi(t_1)}T'')$ and (V_{t_2}, A_{t_2}) is a normal stein space determined by $(M, {}^{\phi(t_2)}T'')$, we should regard

$$(M, {}^{\phi(t_1)}T'') \quad \text{and} \quad (M, {}^{\phi(t_2)}T'')$$

as the same point in the moduli space . However , this equivalence is hardly handled . Because even if $(V_{t_1}, A_{t_1}) \xrightarrow{\sim} (V_{t_2}, A_{t_2})$ as a germ of singularities , we can't say anything about CR-structures . For example , let $M_{a,b}$ be

$$\sum_{i=1}^{n}((x_i^2/a_i^2) + (y_i^2/b_i^2)) = 1 \quad \text{in} \quad C^n \; .$$

Obviously , for any $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, $(M_{a,b}, {}^{o}T'')$ defines the same stein space as the germ of the origin (non-singular point) . However , by Webster (see (W)) ,

$$(M_{a,b}, {}^{o}T'') \xrightarrow{\sim} (M_{a,a}, {}^{o}T'') \quad \text{as a CR-structure}$$

if and only if $a = b$.

We would like to avoid this difficulty . In the above example , we note that Webster's scalar curvature changes if $a \neq b$. Hence we would like to propose a deformation theory of pseudo hermitian CR-structureswhich preserves the Webster's scalar curvature . We must explain what the Webster's scalar curvature is like . Let θ be a real 1 - form satisfying :

$$ {}^{o}T'' + {}^{o}\bar{T}'' = \{ X : X \in C \otimes TM , \theta(X) = 0 \} . $$

We call this triple $(M, {}^{o}T'', \theta)$ a pseudo hermitian structure . By Chern - Moser and Tanaka , we have a connection and curvature forms over the coframe bundle of $(M, {}^{o}T'', \theta)$, like in the case of Riemann geometry , and so we have the Webster's scalar curvature over $(M, {}^{o}T'', \theta)$. We write R for this Webster's scalar curvature . Let $(M, {}^{\phi}T'', \theta)$ be a deformation of the pseudo hermitian structure of $(M, {}^{o}T'', \theta)$. Namely , ϕ is an element of $\Gamma(M, {}^{o}\bar{T}'' \otimes ({}^{o}T'')^*)$ and here we always use the same θ as above . Let $R(\phi)$ be the Webster's scalar curvature defined by $(M, {}^{\phi}T'', \theta)$. We consider

$$ \{ \phi : \phi \in \Gamma(M, {}^{o}\bar{T}'' \otimes ({}^{o}T'')^*) , P(\phi) = 0 , R(\phi) = R \} , $$

where $R(\phi)$ means the Webster'scallar curvature defined by $(M, {}^{\phi}T'', \theta)$. Immediately , we have the following queation . Namely this family has enough deformations or not .

We see this . Let g be a real valued C^∞ function on M . And let X_g be a ${}^\circ\bar{T}''$ - valued vector field defined by :

$$-\theta([X_g, Y]) = Yg \quad \text{for any } Y \text{ in } \Gamma(M, {}^\circ T'') .$$

For any family of deformations $(\mathcal{N}, \pi, S)$, a family of deformations of complex manifold N , where M is a real hypersurface of N , we would like to get a real valued function $g(s)$ satisfying :

there is a C^∞ embedding $f_{X_{g(s)}} : M \hookrightarrow \pi^{-1}(s)$

where

$$f_{X_{g(o)}} = \text{identity}$$

satisfying

$$* \qquad R(\psi(s) \circ f_{X_{g(s)}}) = R \quad .$$

Obviously $*$ is a non-linear partial differential equation , where unknown function is $g(s)$. For this equation , let $L_{\psi(s)}$ be a linear term with respect to g . Then ,

Proposition 1. The principal term of $L_{\psi(s)}$ is sub-elliptic , whivh doesn't depend on $\psi(s)$.

Theorem 2. If

$$\{ g : g \text{ is a real valued } C^\infty \text{ function satisfying } L_o g = 0 \}$$
$$= \{ \text{constant functions} \} ,$$

then

$$\{ \phi : \phi \in \Gamma(M, {}^\circ\bar{T}'' \otimes ({}^\circ T'')^*) , P(\phi) = 0 , R(\phi) = R \}$$

is versal .

Here $\Psi(s)$ means the corresponding element of $\Gamma(M, T' \otimes (^{\rho} T'')^*)$ to $(\mathcal{N}, \pi, S)$.

Details will appear soon in another paper .

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PROPAGATION OF CONVERGENCE ALONG A MOISEZON SUBSPACE

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I. Formal complex spaces (cf.[B],[A-T]).

Let $(X,\mathcal{O}_X)$ be a complex space and S its subspace defined by ideal sheaf $I\subset\mathcal{O}_X$. Then $\hat{\mathcal{O}}_{X,S}:=\varprojlim \mathcal{O}_X/I^k|_{|S|}$ is a sheaf of rings on $|S|$. $\mathbb{C}$-ringed space $\hat{X}\equiv\hat{X}|_S:=(|S|,\hat{\mathcal{O}}_{X,S})$ is called the completion of X along S. Let J be a coherent ideal sheaf of $\hat{\mathcal{O}}_{X,S}$ and put $A:=\mathrm{spt}.\hat{\mathcal{O}}_{X,S}/J\subset|X|$. A $\mathbb{C}$-ringed space locally isomorphic to such $(A,\hat{\mathcal{O}}_{X,S}/J)$ is called a formal complex space. Its structure sheaf is coherent. Its sections are called formal functions. Formal complex spaces and their morphisms as $\mathbb{C}$-ringed spaces form the category of formal complex spaces.

If $S\subset X$, we can canonically define a morphism $\iota_S:\hat{X}|_S\longrightarrow X=\hat{X}|_X$. If $\Phi:X\longrightarrow Y$ is a morphism of complex spaces with $\Phi(|S|)\subset|T|$, there exists a unique morphism $\hat{\Phi}:\hat{X}|_S\longrightarrow\hat{Y}|_T$ such that $\iota_T\circ\hat{\Phi}=\Phi\circ\iota_S$. We call $\hat{\Phi}$ the completion of Φ. A morphism of formal complex spaces is called convergent if it is a completion of a morphism of complex spaces.

II. Results.

Lemma 1 ([I]). Suppose that $S\subset X$ is exceptional and that X is irreducible around S. If global section $f\in\Gamma(S,\hat{\mathcal{O}}_{X,S})$ is convergent at $\xi\in S$, f is convergent along whole S i.e. $f\in\Gamma(S,\mathcal{O}_X)$.

This follows from a corollary of Gabrielov's theorem [G] (cf.[I],[T]).

Theorem 2. Suppose that $S \subset X$ is a (compact irreducible) Moisezon subspace and that X is irreducible around S. If global section $f \in \Gamma(S, \hat{O}_{X,S})$ is convergent at $\xi \in S$, f is convergent along whole S i.e. $f \in \Gamma(S, O_X)$.

This follows from Lemma 1 and the theory of positive line bundles and the complex tubular neighbourhood theorem of Grauert [Gra].

Corollary 3.([T]). Suppose that S is a compact irreducible Moisezon subspace and $D \subset S$ is a Cartier divisor. If A, $B \subset |S| - |D|$ are compact sets with interior, then there exists $M > 0$ such that $\|f\|_B \leqq M^d \|f\|_A$ for any $f \in L(dD)$, where $\| \ \| :=$(the maximum norm) and

$$L(dD) := \{f: \text{ meromorphic on } S,\ (f) + dD \text{ is effective}\}.$$

Theorem 4. Suppose that

(1) $S \subset X$ is a (compact irreducible) Moisezon subspace such that X is irreducible around S;

(2) $T \subset Y$ is a complex subspce;

(3) There exists a finite succession $\Pi: Y \longrightarrow Z$ of finite surjective morphisms and modifications of complex spaces such that $\Pi(|T|)$ is one point;

(4) $\theta: \hat{X}|_S \longrightarrow \hat{Y}|_T$ is a morphism of formal complex spaces.

Then, if θ is convergent at $\xi \in S$, θ is convergent along whole S.

This follows from Theorem 2 and Lemma 6 below.

Lemma 5. Let $\Pi: Y \longrightarrow Z$ be a finite succession of finite surjective morphisms and modifications of complex spaces such that $\Pi(|S|) \subset |T|$. Then there exists a thin subspace $W \subset Z$ depending only on Π such that the following holds.

(1) If $K: \hat{C}|_{(0)} \longrightarrow \hat{Z}|_T$ is a formal curve which is not contained in W, it has only a finite number of liftings $\Lambda: \hat{C}|_{(0)} \longrightarrow \hat{Y}|_T$.

(2) If K is convergent, Λ are also so.

(2) follows from (1) and Artin's theorem [A] on analytic equations.

Lemma 6. Let $\hat{X}|_S$, $\hat{Y}|_T$, $\hat{Z}|_R$ be completions of complex spaces along subspaces. Suppose that $\theta:\hat{X}|_S \longrightarrow \hat{Y}|_T$ is a formal morphism and that $\hat{\Pi}:\hat{Y}|_T \longrightarrow \hat{Z}|_R$ is the completion of a finite succession $\Pi:Y|_T \longrightarrow Z|_R$ of finite surjective morphisms and modifications with $\Pi(|T|) \subset |R|$). If $\hat{\Pi}\circ\theta$ is convergent, so is θ.

This follows from Lemma 5 and a theorem of [L-M] on convergence along lines through $0 \in C^n$.

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Some remarks on the construction of regular algebraic surfaces of general type.

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§1. Motivation

Let S be a minimal algebraic surface of general type defined over the complex number field $\mathbb{C}$. Denote by $p_g(S)$, $c_1{}^2(S)$ the geometric genus and the square of the first Chern class of S respectively. By Noether and Miyaoka's inequality, we have $2p_g(S) - 4 \leq c_1{}^2(S) \leq 9p_g(S) + 9$, $p_g(S) \geq 0$ and $c_1{}^2(S) > 0$. Now we consider the "type" of the canonical mapping Φ_K of S. The following proposition is easily proved by the same argument as in Beauville [B].

(1.1) Proposition. Let S be a minimal algebraic surface of general type.

(1) Assume that Φ_K is generically finite. If there exists an integer i such that $c_1{}^2(S) \leq (i + 1)(p_g(S) - 2)$, then the mapping degree of Φ_K does not greater than i.

(2) Assume that Φ_K gives a pencil of curves of genus g. And assume that there exist integers j and k with $0 \leq k \leq j$ such that $c_1{}^2(S) < (j + 1)(p_g(S) - 1)$ and $p_g(S) > (j/(k+1)) + 1$. Let $|L|$ be the variable part of the canonical system $|K_S|$. Then the number of the base points of $|L|$ is not greater than k, and

$g \leq [(1/2)(j + k)] + 1$ (where $[(1/2)(j + k)]$ is the greatest integer not exceeding $(1/2)(j + k)$).

By the above proposition, for any algebraic surface of general type, the number of the type of Φ_K is finite. For instance:

(1.2) Corollary. Assume that $c_1^2(S) < 6p_g(S) - 6$ and $p_g(S) > 6$. Then one of the following two conditions is satisfied.

(1) Φ_K is generically finite and $\deg \Phi_K \leq 5$.

(2) Φ_K is a holomorphic map and Φ_K gives a pencil of curves of genus 2 or 3 .

Proposition (1.1) is not a deep result, because it does not imply the following well-known facts.

(1.3) Facts. (1) (Beauville [B]) If $c_1^2(S) < 3p_g(S) - 7$, then Φ_K is a generically finite $2 : 1$ mapping.

(2) (Horikawa [H2, §1]) If $c_1^2(S) \leq 4p_g(S) - 7$, $p_g(S) \geq 5$ and $q(S) = 0$, then Φ_K does not give a pencil.

Now we prepare the following definition, and consider the next problem:

(1.4) Definition. We call that S is of type I (resp. of type II) when Φ_K is a birational map (resp. a gererically finite $2 : 1$ map). We call that S is of type ∞ when S is neither of type

I nor of type II.

(1.5) Problem of the geography of surfaces of "fixed type" .
Construct "many" examples of surfaces of type I (resp. of type II or of type ∞).

The geography of surfaces of type II is studied by Persson [P], Xiao [X] and Chen [C] etc. They used the double covering method and the obtained surfaces have pencils of hyperelliptic curves. On the other hand, "a little" examples of surfaces of type ∞ are known ([B, §2],[P, §3],[H2, §4] etc.).

Now our aim in this note is to study the geography of regular surfaces of type I . ("regular" means that the irregularity vanishes.) In §2, we study divisors on $\mathbb{P}^1$-bundles over a Hirzebruch surface. In §3, we study singularities on surfaces. In §4, we construct some examples of surfaces of type I . The obtained surfaces have pencils of nonhyperelliptic curves of genus 3 .

The basic idea in this note was born in usual discussion with Kazuhiro Konno. And also the results in §3 are in the joint work with him [AK2]. The author would like to express hearty thanks to him.

§2. Divisors on P^1-bundles over a Hirzebruch surface

(2.1) Let Σ_e be a Hirzebruch surface of degree e , and let C_0 , f be a 0-section ($C_0^2 = e$) and a fiber on Σ_e respectively. For

integers α and β, let $\pi : X = \mathbb{P}(\mathcal{O}_{\Sigma_e} \oplus \mathcal{O}_{\Sigma_e}(\alpha C_0 + \beta f)) \longrightarrow \Sigma_e$ be the $\mathbb{P}^1$-bunble associated with the locally free sheaf $\mathcal{O}_{\Sigma_e} \oplus \mathcal{O}_{\Sigma_e}(\alpha C_0 + \beta f)$. Set $T = \mathcal{O}_X(1)$, $D_0 = \pi^* C_0$ and $F = \pi^* f$. Then the Picard group of X is generated by T, D_0 and F.

Let x, y, z be integers with $x \geq 2$. We consider an irreducible divisor V on X which is linearly equivalent to $xT + yD_0 + zF$. The restriction map $\pi_{|V} : V \longrightarrow \Sigma_e$ is a generically finite $x : 1$ mapping. And the general fiber V_t of the composition map $\bar{\pi} : V \longrightarrow \Sigma_e \longrightarrow \mathbb{P}^1$ is a curve of arithmetic genus $g(V_t) = (1/2)(x - 1)(x\alpha + 2(y - 1))$. Let $\chi(\mathcal{O}_V)$, $\omega_V{}^2$ be the Euler-Poincare characteristic of $\mathcal{O}_V$ and the self-intersection number of the dualizing sheaf ω_V of V. Then ;

(2.2) Proposition. Put $N = (1/24)x(x - 1)(x - 2)\alpha(\alpha e + 2\beta)$, $H = (1/2)g(((g/(x-1)) + 1)e + x\beta + 2z)$, $p_N = 6(x - 2)/(x - 1)$ and $p_H = (2(3x-4)/(x-1)) - (2x/g)$. Then we have

$$\chi(\mathcal{O}_V) = N + H - (g - 1),$$
$$\omega_V{}^2 = p_N N + p_H H - 8(g - 1).$$

(2.3) Remark. (1) Put $x = 2$. Then we have $N = p_N = 0$ and $p_H = 4(g - 1)/g$. The general fiber of $\bar{\pi} : V \longrightarrow \mathbb{P}^1$ is a hyperelliptic curve. On the other hand, for any surface S of general type which has a linear pencil of hyperelliptic curves of genus g, we have $c_1{}^2(S) \geq (4(g-1)/g)(p_g(S) - g)$. (See Horikawa [H3]). For this reason, we call p_H *the hyperelliptic coefficient* of V.

(2) Let E be a rank 3 vector bundle on a nonsingular curve C,

and let $\pi' : Y = \mathbb{P}(E) \longrightarrow C$ be the $\mathbb{P}^2$-bundle associated with E. Set $T' = \mathcal{O}_Y(1)$. For $x' \geq 4$ and a divisor A on C, let V' be an irreducible divisor on Y which is linearly equivalent to $x'T' + \pi'^*A$. The general fiber of $\pi'|_{V'} : V' \longrightarrow C$ is a nonhyperelliptic curve of arithmetic genus $g = (1/2)(x' - 1)(x' - 2)$. Then by a formula of Takahashi [Ta], we have

$$\omega_{V'}{}^2 = (6(x'-3)/(x'-2))\chi(\mathcal{O}_{V'}) + (x'(x'-3)(x'+1)/(x'-2))(g - 1).$$

The coefficient $6(x'-3)/(x'-2)$ is similar to p_N. So we call p_N *the nonhyperelliptic coefficient* of V.

(2.4) Lemma. Let V be as in (2.1). Set $q(V) = h^1(V, \mathcal{O}_V)$.

(1) If $\alpha + y \geq 1$ and $e + \beta + z \geq 1$, then $q(V)$ vanishes.

(2) Assume $x \geq 3$, $y + (x - 2)\alpha \geq 2$ and $z + (x - 2)\beta + e \geq 2$. Then the rational map associated with the dualizing system $|\omega_V|$ of V is a birational map.

(2.5) Lemma. For a divisor $D = xT + yD_0 + zF$ on X, the following are equivalent:

(1) D is ample.

(2) D is very ample.

(3) $x > 0$, $y > 0$, $z > 0$ and $\beta x + z > 0$.

Thus by Lemma (2.4),(2.5) and Bertini's theorem, we can give some examples of surfaces of type I. The invarients of these surfaces are calculated by Proposition (2.2).

§3. Contribution of singularities.

(3.1) In this section, let S be a normal Gorenstein surface. For an isolated singularity ξ on S, let $\tau : S' \longrightarrow S$ be the minimal resolution of ξ. By the spectral sequence $H^p(S, R^q\tau_*\mathcal{O}_{S'}) \Longrightarrow H^{p+q}(S', \mathcal{O}_{S'})$, we have $\chi(\mathcal{O}_{S'}) = \chi(\mathcal{O}_S) - p_g(\xi)$ where $p_g(\xi) = h^0(S', R^1\tau_*\mathcal{O}_{S'})$ is the geometric genus of ξ. Moreover there exists a divisor Z_ξ on S' supported by $\tau^{-1}(\xi)$ such that $\omega_{S'} = \tau^*\omega_S \otimes \mathcal{O}_{S'}(-Z_\xi)$. (See e.g.[R]). Thus we have $\omega_{S'}{}^2 = \omega_S{}^2 + Z_\xi{}^2$. We call $(p_g(\xi) : -Z_\xi{}^2)$ *the type of singularity* ξ.

The explicite formulae for calculating the type of singularities in some special situations are required. For instance, Tomari[To] obtained a certain formula. On the other hand, in case of double points, Horikawa[H1, §2] obtained another formula. His method — the canonical resolution for double coverings — is useful not only for calculating the local contribution of singularities but also controlling all the singularities on a surface globally.

Now we extend this method for cyclic triple coverings. (See [AK2, §3]). Let W be a nonsingular surface, and let L be a line bundle on W. Set $\pi : X = \mathbb{P}(\mathcal{O}_W \oplus \mathcal{O}_W(L)) \longrightarrow W$ the $\mathbb{P}^1$-bundle, and set $T = \mathcal{O}_X(1)$. If we fix a system of fiber coordinate $(Y_0 : Y_1)$ of $\pi : X \longrightarrow W$, then any section $\varphi \in H^0(X, \mathcal{O}(3T))$ can be written as

$$\varphi = Y_0{}^3 + \varphi_L Y_0{}^2 Y_1 + \varphi_{2L} Y_0 Y_1{}^2 + \varphi_{3L} Y_1{}^3 ,$$

where $\varphi_{iL} \in H^0(W, \mathcal{O}(iL))$, $1 \le i \le 3$. We set

$$|3T|_{\mathscr{C}} = \{ (\varphi) \in |3T| \ ; \ \varphi = Y_0{}^3 + \varphi_{3L} Y_1{}^3 \},$$

where (φ) is the divisor determined by φ. We call it the *cyclic subsystem* of $|3T|$. (This situation is natural in the sence of Wavrik [W, Theorem 1.2]). We call the divisor $B_\varphi = (\varphi_{3L})$ the branch locus of (φ).

For a member $S = S_0$ of $|3T|_{\mathscr{C}}$, we can define a commutative diagram (3.1.1) inductively which satisfies the following properties:

(3.1.1)
$$\begin{array}{ccccccc} S_n & \xrightarrow{\mu_n} & \cdots\cdots\cdots & S_1 & \xrightarrow{\mu_1} & S_0 & = S \\ \downarrow \pi_n & & & \downarrow \pi_1 & & \downarrow \pi_0 & \\ W_n & \xrightarrow[\tau_n]{} & \cdots\cdots\cdots & W_1 & \xrightarrow[\tau_1]{} & W_0 & = W \end{array}$$

(1) For each i, $1 \leq i \leq n$, $\tau_i : W_i \longrightarrow W_{i-1}$ is the blow up of W_{i-1} at a singular point P_i of the branch locus B_{i-1} of π_{i-1} : $S_{i-1} \longrightarrow W_{i-1}$.

(2) Set $L_i = \tau_i{}^*L_{i-1} \otimes \mathcal{O}(-[m_i/3]E_i)$ where m_i is the multplicity of B_{i-1} at P_i and $E_i = \tau_i{}^{-1}(P_i)$. Put $X_i = \mathbb{P}(\mathcal{O}_{W_i} \oplus \mathcal{O}_{W_i}(L_i))$ and $T_i = \mathcal{O}_{X_i}(1)$. Then S_i is a member of $|3T_i|_{\mathscr{C}}$ on X_i. π_i : $S_i \longrightarrow W_i$ is the restriction of the natural projection $X_i \longrightarrow W_i$.

(3) A natural birational morphism $\mu_i : S_i \longrightarrow S_{i-1}$ exists.

(4) The reduced part of B_n is nonsingular, and the singular locus of S_n is of the form $\xi^3 + x^2 = 0$. We call it the *compound cusp*.

Let $\bar{\mu} : S^* \longrightarrow S_n$ be the normalization of S_n. Then S^* is nonsingular. We call $\tilde{\mu} = \bar{\mu} \cdot \mu_n \cdots \mu_1 : S^* \longrightarrow S$ the *canonical resolution*. In general, this is not the minimal resolution. Thus we get the minimal resolution $\tilde{S}$ of S by contracting all (possibly infinitely near) (-1)curves on S^*.

We give the fomrula for calculating the difference between $(\chi(\mathcal{O}_S), \omega_S{}^2)$ and $(\chi(\mathcal{O}_{S^*}), \omega_{S^*}{}^2)$.

(3.2) Proposition. Let $S^* \longrightarrow S$ be the canonical resolution as above. Then,

(1) $\chi(\mathcal{O}_{S^*}) = \chi(\mathcal{O}_S) - \sum_i (1/2)[m_i/3](5[m_i/3]-3) + \sum_j (1-g(C_j)-C_j^2)$,

(2) $\omega_{S^*}^2 = \omega_S^2 - \sum_i 3(2[m_i/3]-1)^2 + \sum_j 8(1-g(C_j))$,

where m_i is the multiplicity of B_{i-1} at the center of the blow up $\tau_i : W_i \longrightarrow W_{i-1}$, $g(C_j)$ and C_j^2 are respectively the genus and the self-intersection number in S^* of the nonsingular curve C_j which is a reduced component of the pull-back by $\bar{\mu}$ of the singular locus of S_n .

(3.3) Examples. By means of our canonical resolution, we resolve some isolated singularities P on a surface S, whose local equations are of the form $\xi^3 + f(x, y) = 0$. For the convenience of the reader, we explain by drawing figures. The type of the singularity is also indicated.

We use the following notations in the figures: The number near the line is the self-intersection number of the curve.
On W_i : the line ——— is a component of the reduced branch locus; the double line ═══ is a component of the double branch locus and the dotted line ---- is an exceptional curve which is not a component of the branch locus.
On S^* and $\tilde{S}$: the line ——— is an exceptional rational curve; the double line ═══ is an exceptional elliptic curve and the dotted line ---- is the inverse image of the branch locus.

Example 1: $\xi^3 + xy = 0$ (a RDP of type A_2); Type (0: 0).

(See Fig. 1)

Example 2: $\xi^3 + x^3 + y^3 = 0$ (a simple elliptic singularity of type

Fig. 1.

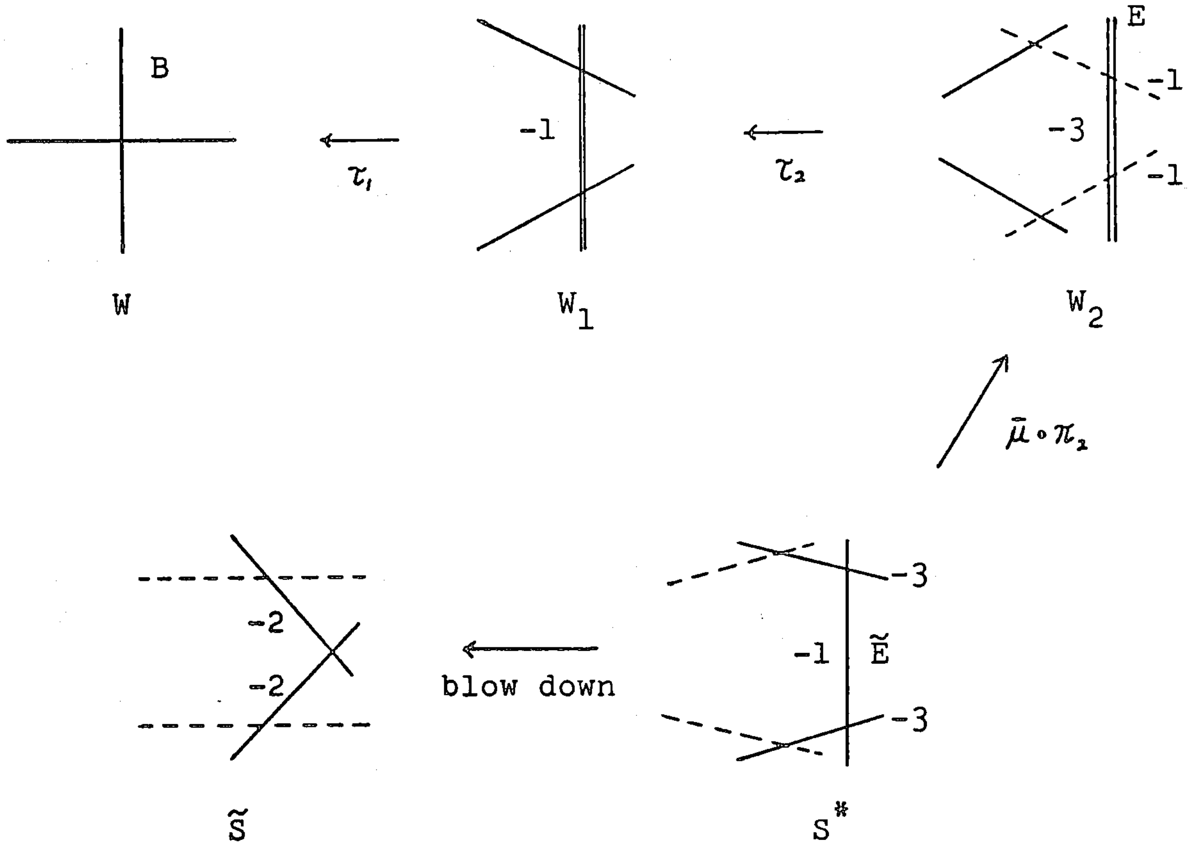

B
-1
-3
E
-1
-1
W
W_1
W_2
τ_1
τ_2
$\bar{\mu}\circ\pi_2$
-2
-2
blow down
-1
$\tilde{E}$
-3
-3
$\tilde{S}$
S^*

Fig. 2.

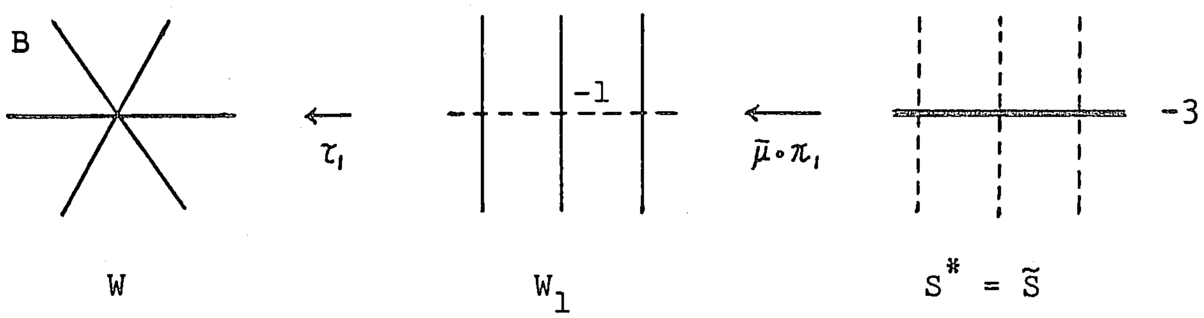

B
-1
-3
τ_1
$\bar{\mu}\circ\pi_1$
W
W_1
$S^* = \tilde{S}$

Fig. 3.

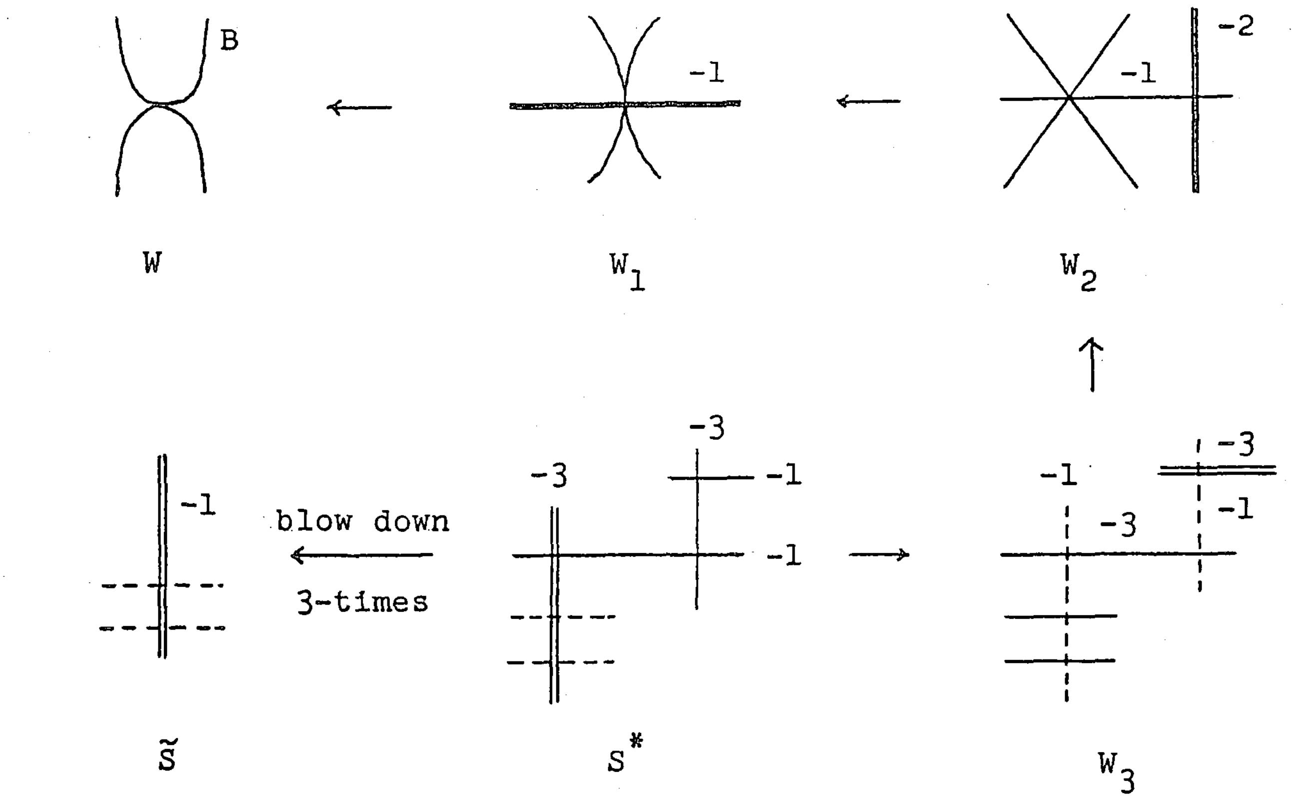
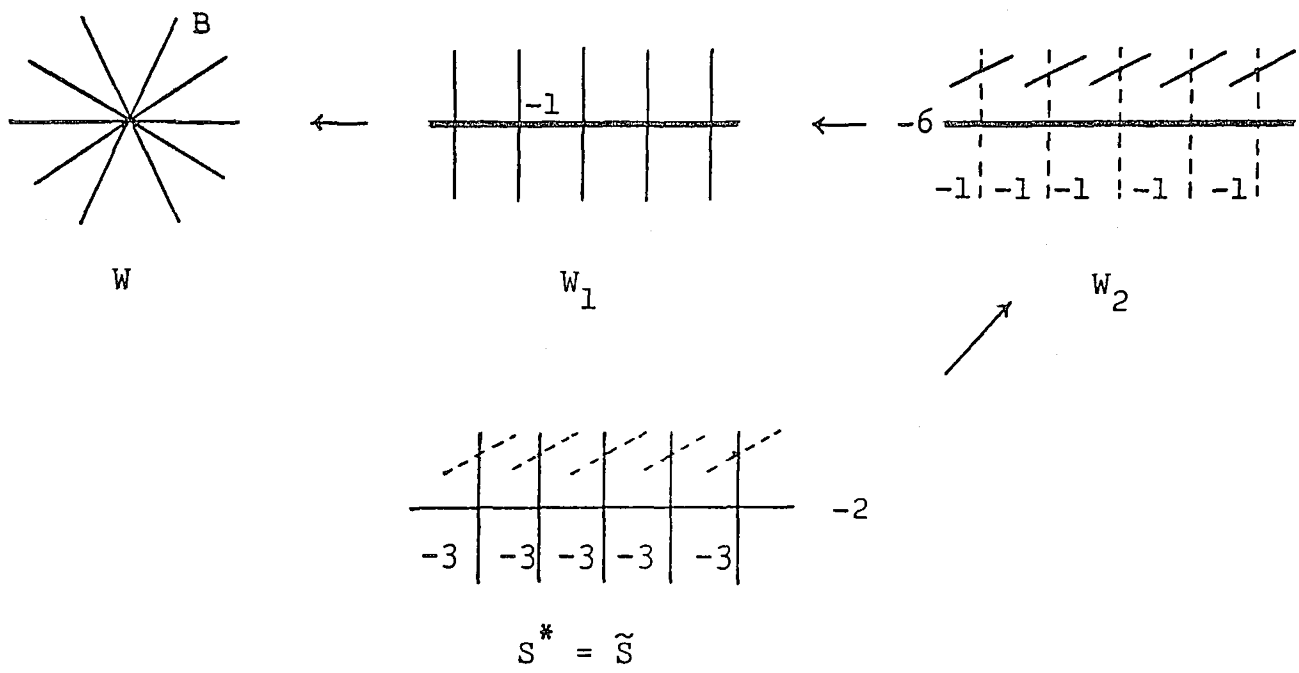

B
-1
-2
-1
W
W1
W2
-3
-3
-1
-1
blow down
3-times
-1
-1
-3
-3
-1
-1
S̃
S*
W3

Fig. 4.


B
-1
-6
-1 -1 -1 -1 -1
W
W1
W2
-2
-3 -3 -3 -3 -3
S* = S̃

$\tilde{E}_6$, see [S]); Type (1: 3).

(Fig. 2)

Example 3: $\xi^3 + x^2 + y^6 = 0$ (a simple elliptic singularity of type $\tilde{E}_8$, see [loc. cit]); Type (1: 1).

(Fig. 3)

Example 4: $\xi^3 + x^5 + y^5 = 0$; Type (3: 10). Compare this with Tomari [To, (2.9)].

(Fig. 4)

(3.4) ***Problem.*** Extend the method of canonical resolution to more general case.

We remark that the canonical resolution for cyclic quadruple covering in some special case is used in the next section.

§4. Construction of surfaces of type I .

In this section, we construct some examples of regular algebraic surfaces of type I.

(4.1) Let ℓ, m, n be integers satisfying

(4.1.1) $0 \le \ell \le m$, $\ell + m + n \ge 2$, $4\ell + n \ge 0$, $m + n \ge 0$,

$(\ell, m, n) \ne (0, 0, 2)$.

Let $X = \mathbb{P}(\mathcal{O}_{\Sigma_{m-\ell}} \oplus \mathcal{O}_{\Sigma_{m-\ell}}(C_0 + \ell f))$ be the $\mathbb{P}^1$-bundle over the Hirzebruch surface $\Sigma_{m-\ell}$ of degree $m - \ell$. By (4.1.1), there exists an irreducible normal member S_1 of the complete linear system $|4T + nF|$ from the argument in [AK1, §2]. Then we have

$$\chi(\mathcal{O}_{S_1}) = 4(\ell + m) + 3n - 2 ,$$
$$\omega_{S_1}{}^2 = 12(\ell + m) + 8n - 16 ,$$

by Proposition (2.2). It is easy to see that S_1 has n exceptional curves of first kind. Let $S_1 \longrightarrow S_0$ be the contraction of these curves. Then we have

$$\chi(\mathcal{O}_{S_0}) = \chi(\mathcal{O}_{S_1}) ,$$
$$\omega_{S_0}{}^2 = 12(\ell + m) + 9n - 16 = 3\chi(\mathcal{O}_{S_0}) - 10 .$$

A calculation show that $q(S_0) = 0$. If S_0 is nonsingular, S_0 is called a *Castelnuovo surface* of type (ℓ, m, n). (See [AK1, §4]).

Now we construct a normal member S_1 of $|4T + nF|$ such that any isolated singularity of S_1 is either simple elliptic singularity of type $\tilde{E}_7$ (see [S]) or R.D.P. of type A_3.

Let $(Y_0 : Y_1)$ be a fiber coordinate of $X \longrightarrow \Sigma_{m-\ell}$ and fix it. We define the *good cyclic system* $|4T + nF|_{\mathcal{GC}}$ of $|4T + nF|$ by the following:

(1) Any member φ of $|4T + nF|_{\mathcal{GC}}$ is written as

$$\varphi = \varphi_{nf}\, Y_0{}^4 + \varphi_{4C_0+(4\ell+n)f}\, Y_1{}^4 ,$$

where φ_{nf} and $\varphi_{4C_0+(4C_0+n)f}$ are members of the linear system $|nf|$ and $|4C_0 + (4\ell + n)f|$ on $\Sigma_{m-\ell}$ respectively.

(2) The branch locus $B_\varphi := (\varphi_{4C_0+(4\ell+n)f})$ is reduced.

(3) The divisor $B_\varphi' := (\varphi_{nf})$ is reduced and does not meet the singular locus of B_φ. We call it the *assistant branch locus* of (φ).

We construct a member S_1 of $|4T + nF|_{\mathcal{GC}}$ by the following;

Let β, k_0, k_∞ be nonnegative integers satisfying

(4.1.2) $\quad k_0 \le 2(m - \ell) + \beta , \quad k_\infty \le \beta , \quad 4\ell + n - 2\beta \ge 0 .$

Denote by C_∞ the ∞-section of $\Sigma_{m-\ell}$. We fix k_0 general points $P_1, \cdots, P_{k_0}$ on C_0, and fix k_∞ general points $P_{k_0+1}, \cdots, P_{k_0+k_\infty}$ on C_∞. Then by (4.1.2), we can choose members B^i ($i = 1, 2$) of Persson's subsystem $|2C_0 + \beta f|_{\mathcal{P}}$ (see [AK1, §6]) such that B^i ($i = 1, 2$) pass through $P_1, \cdots, P_{k_0+k_\infty}$. We put

$$B := B^1 + B^2 + (4\ell + n - 2\beta)f.$$

B is linearly equivalent to $4C_0 + (4\ell + n)f$. Put $k = k_0 + k_\infty$. Then we have $k \le 2(m - \ell) + 2\beta \le 4(m + n) + n$. We assume that B is a reduced divisor such that (1) B has k infinitely close double points, and (2) other singularities of B are at most ordinary double points. Let B' be n distinct fibers which does not pass through $P_1, \cdots, P_{k_0+k_\infty}$. Then we can define a member S_1 of $|4T + nF|_{\mathcal{G}\mathcal{G}}$ whose branch and assistant branch respectively are B and B'. S_1 has k $\tilde{E}_7$-singularities and other singularities are R.D.P. of type A_3.

Let $\tilde{S} \longrightarrow S_1$ be the minimal resolution of singularities of S_1. Since $\tilde{E}_7$ is of type (1: 2) in the sence of (3.1), we have

$$\chi(\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{S_1}) - k, \quad \omega_{\tilde{S}}^2 = \omega_{S_1}^2 - 2k.$$

Now we have the following lemma:

(4.2) Lemma. If $k \le 2\ell + m + n - 2$, then $\tilde{S}$ is a minimal regular surface of type I.

For the proof of this lemma, the method of canonical resolution of cyclic quadruple covering is applied. From the above argument, we have the following proposition:

(4.3) Proposition. Let x, y be any integers satisfying $3x - 7 \leq y \leq (18/5)x - (28/5)$ and $x \geq 4$. Then there exists a minimal algebraic surface S of general type such that

(1) $p_g(S) = x$, $c_1{}^2(S) = y$ and $q(S) = 0$,

(2) the canonical map Φ_K of S induces a birational map,

(3) S has a linear pencil of nonhyperelliptic curves of genus 3.

(4.4) Remarks. (1) Let S be a surface such that Φ_K of S is birational. Then by the Castelnuovo's second inequality ([BPV,p228]) we have $c_1{}^2(S) \geq 3p_g(S) - 7$. The surfaces with $c_1{}^2 = 3p_g - 7$ are studied in [AK1].

(2) We consider surfaces which have pencils of nonhyperelliptic curves of genus ≥ 4. By Proposition (2.2) and the similar method as in §4, it seems that one can obtain "many" regular surfaces of type I.

(3) Let Σ be a geometrically ruled surface of genus ≥ 1. Let X be a $\mathbb{P}^1$-bundle over Σ. Then by the similar method as in this note, one can obtain *irrugular surfaces* of type I. The obtained surfaces in this method cover "very wide area" in the domain $2\chi - 6 \leq c_1{}^2 \leq 9\chi$.

The detail will be published in our forthcoming paper.

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Hypersurface sections of toric singularities

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Introduction. We can obtain much information about hypersurface singularities $\{ f = 0 \}$ in $\mathbb{C}^{n+1}$ by the Newton polyhedra $\Gamma_+(f) \subset \mathbb{R}^{n+1}$ of the defining equations f. (For instance, see [5] and [11].) Also for hypersurface sections (X, x) of a toric singularity (Y, y), we can define the Newton polyhedra and almost all of the results in [5] and [11] are valid. On the other hand, we obtain as (X, x) many singularities, a part of which are not complete intersection, such as 2-dimensional cusp singularities with multiplicities greater than 4 and a 3-dimensional singularity with a resolution whose exceptional set is an Enriques surface. Moreover, we easily see what singularities appear in small deformations of X which are still hypersurface sections of Y. For instance, if the ambient space Y has only an isolated singularity, then these singularities (X, x) are smoothable. Hence we can obtain examples of smoothable cusp singularities. In this paper, we are mainly concerned about singularities (X, x) with the plurigenera $\delta_m(X, x)$ which are not greater than 1 and at least one of which is equal to 1. (For the definition of plurigenera, see [11].) We call such singularities, *periodically elliptic singularities*, following Ishii [2].

In Section 1, we recall some facts about toric singularities, necessary in this paper.

In Section 2, we show a sufficient condition on the Newton polyhedra of defining equations f of X, under which (X, x) are periodically elliptic singularities and give some examples.

In Section 3, we show a sufficient condition on a 3-dimensional non-terminal Gorenstein toric singularity (Y, y), under which hyperplane sections (X, x) of (Y, y) are simple elliptic singularities or cusp singularities. We can determine the multiplicities of these singularities.

In Section 4, we show that if $H^1(X \setminus \{x\}, i^*\Theta_Y) = 0$ and $\dim X \geqq 3$, then we can concretely construct a locally semiuniversal family of deformations of (X, x) and that any small deformation of (X, x) is also a hypersurface section of Y, where $i : X \hookrightarrow Y$ is the inclusion map and Θ_Y is the tangent sheaf of Y. The above condition is satisfied, if Y is a quotient of $\mathbb{C}^{n+1}$, by torus actions.

We use the notation and the terminology in [4] freely.

I would like to thank Professor M. Tomari who pointed out me the facts that hypersurface sections (X, x) of toric singularities (Y, y) are Cohen-Macaulay and that (X, x) are smoothable, if (Y, y) is an isolated singularity.

§1 Toric singularities

Let N be a free $\mathbb{Z}$-module of rank $n+1$ and let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $M = \mathrm{Hom}(N, \mathbb{Z})$ be the $\mathbb{Z}$-module dual to N with the canonical pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$. Let $\sigma = \mathbb{R}_{\geq 0} u_1 + \mathbb{R}_{\geq 0} u_2 + \ldots + \mathbb{R}_{\geq 0} u_s$ be an $(n+1)$-dimensional strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. Let Y be the complex space associated to $\mathrm{Spec}(\mathbb{C}[M \cap \sigma^*])$ and let $e(v) : Y \to \mathbb{C}$ be the natural extension to Y of the character $v \otimes 1_{\mathbb{C}^\times} : T_N \to \mathbb{C}^\times$ for each v in $M \cap \sigma^*$, where σ^* is the dual cone of σ and $T_N = \mathrm{Spec}(\mathbb{C}[M])$ ($\simeq (\mathbb{C}^\times)^{n+1}$). Then any holomorphic function f on a neighborhood U of $y = \mathrm{orb}(\sigma)$ ($= \{ x \in Y \mid e(v)(x) = 0$ for any $v \in (\sigma^* \setminus \{0\}) \cap M \}$) is expressed as the series:

$$f = \textstyle\sum_{v \in \sigma^* \cap M} c_v e(v).$$

Hence we can define the Newton polyhedron $\Gamma_+(f)$ and the Newton boundary $\Gamma(f)$ of f in the same way as in the case of $Y = \mathbb{C}^{n+1}$. Namely, $\Gamma_+(f)$ is the convex hull of $\bigcup_{c_v \neq 0} v + \sigma^*$ and $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$. Let $D = D_1 + D_2 + \ldots + D_s$, where D_i is the closure of $\mathrm{orb}(\mathbb{R}_{\geq 0} u_i)$ ($= \{ x \in Y \mid e(v)(x) = 0$ for any $v \in \sigma^* \cap M$ with $\langle v, u_i \rangle > 0 \}$). Here we note that $Y \setminus D = T_N$ and that Y is a Cohen-Macaulay space by [4, Corollary 3.9]. Let $\{v_1, v_2, \ldots, v_{n+1}\}$ be a basis of M and let $w_i = e(v_i)$ for $i = 1$ through $n+1$. Then $(w_1, w_2, \ldots, w_{n+1})$ is a global coordinate of T_N. Let $\nu = (dw_1/w_1) \wedge (dw_2/w_2) \wedge \ldots$

$\wedge(dw_{n+1}/w_{n+1})$. Then ν is a nowhere vanishing holomorphic $(n+1)$-form on T_N whose natural extension to Y has poles of order 1 along D.

Definition 1.1. (Y, y) is said to be r-Gorenstein, if there exists a nowhere vanishing holomorphic r-ple $(n+1)$-form on $U \setminus \mathrm{Sing}(U)$ for an open neighborhood U of y, where $\mathrm{Sing}(U)$ is the singular locus of U.

Since (Y, y) is a Cohen-Macaulay singularity, (Y, y) is Gorenstein, if it is 1-Gorenstein.

Proposition 1.2. (Y, y) is r-Gorenstein, if and only if there exists an element v_0 in $M_{\mathbb{Q}}$ such that $rv_0 \in M$ and that $\langle v_0, u_i \rangle = 1$ for $i = 1$ through s, where we assume that u_1, u_2, ... and u_s are primitive elements in N.
(Here we note that the above v_0 is uniquely determined by σ, if it exists.)

Proof. Let v_0 be an element in $M_{\mathbb{Q}}$ satisfying the above condition. Then $\theta := e(rv_0)\nu^r$ is a nowhere vanishing holomorphic r-ple $(n+1)$-form on $Y \setminus \mathrm{Sing}(Y)$, because $e(rv_0)$ has zeros of order $\langle rv_0, u_i \rangle = r$ only along D. Conversely, assume that (Y, y) is r-Gorenstein, i.e., there exists a nowhere vanishing holomorphic r-ple $(n+1)$-form θ on

$U \setminus \mathrm{Sing}(U)$ for an open neighborhood U of y. Then $f := \theta/\nu^r$ is a holomorphic function on $U \setminus \mathrm{Sing}(U)$ which does not vanish on $T_N \cap U$ and whose vanishing order at D_i is equal to r. Since the codimension of $\mathrm{Sing}(Y)$ is greater than 1, f is extended to U, by [1, Chapter II, Corollary 3.12]. Hence f is expressed as the series $\sum_{v \in (\sigma^* \setminus \{0\}) \cap M} c_v e(v)$. Suppose that $\Gamma_+(f)$ has a compact face Δ with $\dim \Delta \geqq 1$. Then there exist a primitive element u_0 in $\mathrm{Int}(\sigma) \cap N$ and a positive integer t such that $\langle v, u_0 \rangle = t$ (resp. $> t$) for any element v in Δ (resp. $\Gamma_+(f) \setminus \Delta$). Let Y_0 be the complex space associated to $\mathrm{Spec}(\mathbb{C}[(\mathbb{R}_{\geqq 0} u_0)^* \cap M])$ and let $D_0 = \mathrm{orb}(\mathbb{R}_{\geqq 0} u_0)$. Then we have a holomorphic map $\pi : Y_0 \to Y$ such that $\pi_{|T_N} = \mathrm{id}$ and that $\pi^{-1}(y) = D_0$, because $\mathbb{R}_{>0} u_0 \subset \mathrm{Int}(\sigma)$. Take a basis $\{v'_1, v'_2, \ldots, v'_{n+1}\}$ of M so that $\langle v'_1, u_0 \rangle = 1$ and that $\langle v'_i, u_0 \rangle = 0$ for $i = 2$ through $n+1$. Let $z_i = e(v'_i)$ for $i = 1$ through $n+1$. Then $D_0 = \{ z_1 = 0 \}$ and $f = z_1^t g_0 + z_1^{t+1} g_1 + \ldots + z_1^{t+i} g_i + \ldots$ on $U \cap T_N$, where $g_i = \sum_{v \in L_i} c_v e(v - (t+i) v'_1)$ and $L_i = \{ v \in \Gamma_+(f) \cap M \mid \langle v, u_0 \rangle = t + i \}$. Here we note that g_i are polynomials with variables $z_2, \ldots, z_{n+1}$ and that $g_0 = \sum_{v \in \Delta \cap M} c_v e(v - t v'_1)$ is not a monomial, because the cardinal number of $\{ v \in \Delta \cap M \mid c_v \neq 0 \}$ is greater than 1. Hence $\{ y' \in U \cap T_N \mid (g_0 + z_1 g_1 + \ldots)(y') = 0 \} \neq \emptyset$, because $Y \setminus D_0 = T_N$. Then f must vanish at a point of $U \cap T_N$, a contradiction. Therefore, any compact face of $\Gamma_+(f)$ is a point. This implies that $\Gamma(f)$

consists of only one point v_0'. Hence $\Gamma_+(f) = v_0' + \sigma^*$. Therefore, $<v_0', u_i> \leqq <v, u_i>$ for any element v in $\Gamma_+(f) \cap M$ and for $i = 1$ through $n+1$. Since the vanishing order of f at D_i is r, we have $<v_0', u_i> = r$. Hence the point $v_0 = \frac{1}{r}v_0'$ satisfies the condition of the proposition. q.e.d.

Remark. If $N = \mathbb{Z}^{n+1}$ and $\sigma = (\mathbb{R}_{\geq 0})^{n+1}$, then Y is isomorphic to $\mathbb{C}^{n+1}$ and the point y corresponds to the origin. Clearly $v_0 = (1, 1, \ldots, 1)$ satisfies the condition of the above proposition, if we identify M with N, by the canonical inner product.

§2 Hypersuface sections

Let f be an element of the maximal ideal $\mathfrak{m}_{Y,y}$ of Y at y, let $X = \{ f = 0 \}$ and let $x = y$. Throughout the rest of this paper, we assume that X is irreducible reduced, that (X, x) is an isolated singularity and that $X \cap \mathrm{Sing}(Y) = \{x\}$. By [1, Chapter I, Proposition 1.6 (ii) and Corollary 4.4], we have:

Proposition 2.1 (X, x) is a Cohen-Macaulay and normal singularity.

Assume that $f = \sum_{v \in (\sigma^* \setminus \{0\}) \cap M} c_v e(v)$ is non-degenerate, i.e.,

$$\partial f_\Delta / \partial w_1 = \partial f_\Delta / \partial w_2 = \dots = \partial f_\Delta / \partial w_{n+1} = 0$$

has no solutions in $T_N = Y \setminus D$ ($\simeq (\mathbb{C}^\times)^{n+1}$), for each face Δ of $\Gamma(f)$, where $f_\Delta = \sum_{v \in \Delta \cap M} c_v e(v)$ and $(w_1, w_2, \dots, w_{n+1})$ is a global coordinate of T_N.

Theorem 2.2. Assume that (Y, y) is r-Gorenstein, (that (Y, y) is not r'-Gorenstein for $1 \leqq r' < r$) and let v_0 be the element satisfying the condition of Proposition 1.2. Then (X, x) is r-Gorenstein. Moreover, if v_0 is on $\Gamma(f)$, then

$$\delta_m(X, x) = \begin{cases} 1 & \text{for } m \equiv 0 \mod r \\ 0 & \text{for } m \not\equiv 0 \mod r. \end{cases}$$

(See [11], for the definition of $\delta_m(X, x)$.)

For the proof, we need some preparations. For $u \in \sigma$, let $d(u) = \min \{ \langle v, u\rangle \mid v \in \Gamma_+(f) \}$ and let $\Delta(u) = \{ v \in \Gamma_+(f) \mid \langle v, u\rangle = d(u) \}$. For a face Δ of $\Gamma_+(f)$, let $\Delta^* = \{ u \in \sigma \mid \Delta(u) \supset \Delta \}$. Then $\Gamma^*(f) := \{ \Delta^* \mid \Delta$ is a face of $\Gamma_+(f) \} \cup \{0\}$ is an r.p.p. decomposition of $N_{\mathbb{R}}$ with $|\Gamma^*(f)|$ ($:= \bigcup_{\Delta^* \in \Gamma^*(f)} \Delta^*$) $= \sigma$. Let Σ^* be a subdivision of $\Gamma^*(f)$ consisting of non-singular cones and let $\tilde{Y} = T_N\mathrm{emb}(\Sigma^*)$. Then we have a resolution $\Pi : \tilde{Y} \to Y$ of Y. Let $\tilde{X}$ be the proper transformation of X under Π and let $E = \tilde{X} \cap \Pi^{-1}(x)$. Then π ($:= \Pi_{|\tilde{X}}$) : $\tilde{X} \to X$ is a resolution of X whose exceptional set is E. Assume that u is a primitive element in N and that $\mathbb{R}_{\geqq 0}u$ is a 1-dimensional cone in Σ^* with $\dim \Delta(u) \geqq 1$. Then we denote by $E(u)$ the closure of $\mathrm{orb}(\mathbb{R}_{\geqq 0}u) \cap E$ ($\neq \emptyset$). Recall that $\theta = e(rv_0)\nu^r$ is a nowhere vanishing r-ple (n+1)-form on $Y \setminus \mathrm{Sing}(Y)$. Let $\omega = \mathrm{Res}(\theta/f^r)$, i.e., $\omega = g_{|X\cap U}(dw_1 \wedge \ldots \wedge dw_n)^r$ on $X \cap U$, if θ is expressed as $g(df \wedge dw_1 \wedge \ldots \wedge dw_n)^r$ on an open set U of Y.

Lemma 2.3. $\pi^*\omega^\ell$ has zeros of order $\ell r(\langle v_0, u\rangle - 1 - d(u))$ along $E(u)$.

Proof. The lemma follows from the fact that $e(rv_0)$, ν^r and f^r have zeros of order $r\langle v_0, u\rangle$, $-r$ and $rd(u)$, respectively, along $\mathrm{orb}(\mathbb{R}_{\geqq 0}u)$. q.e.d.

Proof of Theorem 2.2. Since ω is a nowhere vanishing holomorphic r-ple n-form on $X \setminus \{x\}$, we see that (X,x) is r-Gorenstein. Assume that v_0 is on $\Gamma(f)$. Then $\langle v_0, u\rangle \geqq d(u)$ for any u in $\mathrm{Int}(\sigma) \cap N$. Hence the nowhere vanishing holomorphic ℓr-ple n-form $\pi^*\omega^\ell$ has poles of order at most ℓr along each irreducible component of the exceptional set E, by Lemma 2.3. On the other hand, $\Gamma_+(f)$ has a compact face Δ_0 containing v_0 with $\dim \Delta_0 \geqq 1$. Otherwise, $\Gamma_+(f) = v_0 +$

σ^* and hence $f = e(v_0)g$ for a holomorphic function g on Y. Then since $[e(v_0)] = rD$, we get a contradiction to the assumption that X is irreducible. Hence we can take a *subdivision* Σ^* of $\Gamma^*(f)$ so that $\Delta(u_0) = \Delta_0$ for a 1-dimensional cone $\Delta^* = \mathbb{R}_{\geq 0}u_0$ in Σ^*. Then $u_0 \in \mathrm{Int}(\sigma)$, $\mathrm{orb}(\Delta^*) \cap \tilde{X} \neq \emptyset$ and $\langle v_0, u_0\rangle = d(u_0)$. Hence $\pi^*\omega^\ell$ has poles of order ℓr along the irreducible component $E(u_0)$ of E. Therefore, $\delta_{\ell r}(X, x) = 1$. Next, assume that $m \not\equiv 0 \mod r$ and let η be an element in $H^0(X \setminus \{x\}, \mathcal{O}_X(mK_X))$. In the following, we show that η is in $L^{2/m}(X \setminus \{x\})$. Take n elements v_1, v_2, ... and v_n in M so that $\{rv_0, v_1, \dots, v_n\}$ is a basis of M. Let $w_0 = e(rv_0)$ and let $w_i = e(v_i)$ for $i = 1$ through n. Then $(w_0, w_1, \dots, w_n)$ is a global coordinate of T_N. Let $M' = M + \mathbb{Z}v_0$ and let $N' = \{ u \in N \mid \langle v', u\rangle \in \mathbb{Z}$ for any $v' \in M' \}$ ($= \{ u \in N \mid \langle v_0, u\rangle \in \mathbb{Z} \}$). Then the inclusion $N' \to N$ induces a holomorphic map $\varphi : Y' \to Y$, where Y' is the complex space associated to $\mathrm{Spec}(\mathbb{C}[M' \cap \sigma^*])$. Since $\{v_0, v_1, \dots, v_n\}$ is a basis of M', $(z_0, z_1, \dots, z_n)$ is a global coordinate of $T_{N'} = \mathrm{Spec}(\mathbb{C}[M'])$, where $z_i = e(v_i)$ for $i = 0$ through n. Clearly, $\varphi^*w_0 = (z_0)^r$ and $\varphi^*w_i = z_i$ for $i = 1$ through n. Hence φ is the quotient map under the group $\langle t\rangle$ generated by the element $t = (\xi, 1, \dots, 1)$ in $T_{N'}$, where ξ is a primitive r-th root of 1. Moreover, φ is unramified over $Y \setminus \mathrm{Sing}(Y)$, because $\theta := w_0((dw_0/w_0)\wedge(dw_1/w_1)\wedge \dots \wedge(dw_n/w_n))^r$ (resp. $\theta' :=$

$z_0(dz_0/z_0)\wedge(dz_1/z_1)\wedge \ \ldots \ \wedge(dz_n/z_n)$) is a nowhere vanishing holomorphic r-ple (n+1)-form on $Y \setminus \mathrm{Sing}(Y)$ (resp. (n+1)-form on $Y' \setminus \mathrm{Sing}(Y')$) and $\varphi^*\theta = (r\theta')^r$. Hence the set of the singularities of $X' := \varphi^{-1}(X)$ consists of only one point $x' := \varphi^{-1}(x)$. Let $\omega' = \mathrm{Res}(\theta'/\varphi^* f)$. Then ω' is a nowhere vanishing holomorphic n-form on $X' \setminus \{x'\}$ with $t^*\omega' = \xi\omega'$, because $t^* z_0 = \xi z_0$ and $t^*(dz_i/z_i) = dz_i/z_i$ for $i = 0$ through n. Hence $\varphi^*\eta = g(\omega')^m$ for a holomorphic function g on X'. Since $t^*(\varphi^*\eta) = \varphi^*\eta$ and $t^*(g(\omega')^m) = t^*g\xi^m(\omega')^m$, we have $t^*g = \xi^{-m}g$. Since $\xi^{-m} \neq 1$, we have $g(x') = 0$. Hence $\varphi^*\eta = g(\omega')^m \in L^{2/m}(X' \setminus \{x'\})$, because $(\pi')^*\omega' \in H^0(\tilde{X}', \mathcal{O}(K_{\tilde{X}'}+E'))$, for any resolution $\pi' : (\tilde{X}', E') \to (X', x')$ of (X', x'). Therefore, $\eta \in L^{2/m}(X \setminus \{x\})$. Thus we conclude that $\delta_m(X, x) = 0$. q.e.d.

We can obtain a system of defining equations of X from those of Y and f.

Proposition 2.4. If $f \notin \mathfrak{m}^2_{Y,y}$ (resp. $f \in \mathfrak{m}^2_{Y,y}$), then $\dim \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} = \dim \mathfrak{m}_{Y,y}/\mathfrak{m}^2_{Y,y} - 1$ (resp. $\dim \mathfrak{m}_{Y,y}/\mathfrak{m}^2_{Y,y}$).

Proof. We have the following exact sequence.

$$0 \to f\cdot\mathcal{O}_{Y,y}/(f\cdot\mathcal{O}_{Y,y}\cap\mathfrak{m}^2_{Y,y}) \to \mathfrak{m}_{Y,y}/\mathfrak{m}^2_{Y,y} \to \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} \to 0.$$

We easily see that $\dim f\cdot\mathcal{O}_{Y,y}/(f\cdot\mathcal{O}_{Y,y}\cap\mathfrak{m}_{Y,y}^2) = 1$ or 0, according as $f \notin \mathfrak{m}_{Y,y}^2$ or $f \in \mathfrak{m}_{Y,y}^2$. q.e.d.

Assume that $\sigma^* \cap M$ is generated by m elements v_1, v_2, $\dots$, v_m and let $z_i = e(v_i)$, for $i = 1$ through m. Then we have the embedding $i : Y \ni p \mapsto (z_1(p), z_2(p), \dots, z_m(p)) \in \mathbb{C}^m$. Assume that $i(Y)$ is defined by $g_1(z) = g_2(z) = \dots = g_t(z) = 0$, where $z = (z_1, z_2, \dots, z_m)$. If $f \in \mathfrak{m}_{Y,y}^2$, then $X = \{ f = 0 \}$ is isomorphic to the subvariety in $\mathbb{C}^m$ defined by $\tilde{f}(z) = g_1(z) = \dots = g_t(z) = 0$, where $\tilde{f}(z)$ is a holomorphic function on $\mathbb{C}^m$ with $i^*\tilde{f} = f$. Next, assume that we can express $\tilde{f}(z) = z_1 - h(z_2, \dots, z_m)$. Hence $f \notin \mathfrak{m}_{Y,y}^2$. Then X is isomorphic to the subvariety in $\mathbb{C}^{m-1}$ defined by $g_1'(w) = g_2'(w) = \dots = g_t'(w) = 0$, where $w = (z_2, \dots, z_m)$ and $g_i'(w) = g_i(h(w), z_2, \dots, z_m)$.

Example 1. Let $n = 2$, let $\{u_1, u_2, u_3\}$ be a basis of N and let $\{v_1, v_2, v_3\}$ be the basis of M dual to $\{u_1, u_2, u_3\}$. Let $\sigma = \mathbb{R}_{\geqq 0}(u_1 + u_3) + \mathbb{R}_{\geqq 0}(u_1 + u_2 + u_3) + \mathbb{R}_{\geqq 0}(u_2 + u_3) + \mathbb{R}_{\geqq 0}(-u_1 + u_3) + \mathbb{R}_{\geqq 0}(-u_1 - u_2 + u_3) + \mathbb{R}_{\geqq 0}(-u_2 + u_3)$. Then (Y, y) is Gorenstein and $v_0 = v_3$ satisfies the condition of Proposition 1.2. We see that $\sigma^* \cap M$ is generated by $\ell_0 = v_3$, $\ell_1 = v_1 + v_3$, $\ell_2 = v_2 + v_3$, $\ell_3 = -v_1 + v_2 + v_3$, $\ell_4 = -v_1 + v_3$, $\ell_5 = -v_2 + v_3$ and $\ell_6 = v_1 - v_2 + v_3$. (See Figure 1.) Hence Y is isomorphic to the subvariety in $\mathbb{C}^7$ defined by the equations (1) $z_0z_1 - z_6z_2 = z_0z_2 - z_1z_3 = z_0z_3 - z_2z_4 = z_0z_4 - z_3z_5 = z_0z_5 - z_4z_6 = z_0z_6 - z_5z_1 = z_0^2 - z_1z_4 = z_0^2 - z_2z_5 = z_0^2 - z_3z_6 = 0$, where $z_i = e(\ell_i)$, for $i = 0$ through 6. Let $f = z_0 - z_1^2 - z_2^2 - \dots - z_6^2$. Then (X, x) is a cusp singularity with a resolution $\pi : (\tilde{X}, E) \to (X, x)$ such that the exceptional set E is a cycle of six rational curves whose self-intersection numbers are all -3. Since $f \notin \mathfrak{m}_{Y,y}^2$, we see that X is isomorphic to the subvariety in $\mathbb{C}^6$ defined by the equations obtained from the above equations (1), replacing z_0 by $z_1^2 + z_2^2 + \dots + z_6^2$.

Example 2. Let n, $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ be the same as in Example 1. Let $\sigma = \mathbb{R}_{\geqq 0}(u_1 + 2u_3) + \mathbb{R}_{\geqq 0}(u_2 + 2u_3) + \mathbb{R}_{\geqq 0}(u_1 + 2u_2 + 2u_3) + \mathbb{R}_{\geqq 0}(2u_1 + u_2 + 2u_3)$. Then (Y, y) is 2-Gorenstein and $v_0 = \frac{1}{2}v_3$ satisfies the condition of Proposition 1.2. We see that $\sigma^* \cap M$ is generated by $\ell_1 = -2v_1 - 2v_2 + 3v_3$, $\ell_2 = -v_1 + v_3$, $\ell_3 = -2v_1 + 2v_2 + v_3$, $\ell_4 = v_2$, $\ell_5 = 2v_1 + 2v_2 - v_3$, $\ell_6 = v_1$, $\ell_7 = 2v_1 - 2v_2 + v_3$ and $\ell_8 = -v_2 + v_3$. (See Figure 2.) Let $z_i = e(\ell_i)$ for $i = 1$ through 8 and let $f = z_2 - z_4 + z_6 + z_8$. Then f is non-degenerate, (X, x) is an isolated singularity and $X \cap \mathrm{Sing}(Y) = \{x\}$. Moreover, (X, x) is a quotient of a simple elliptic singularity.

Example 3. Let $n = 3$, let $\{u_1, u_2, u_3, u_4\}$ be a basis of N and let $\{v_1, v_2, v_3, v_4\}$ be the basis of M dual to $\{u_1, u_2, u_3, u_4\}$. Let $\sigma = \mathbb{R}_{\geqq 0}(u_1 + u_2 + 2u_4) + \mathbb{R}_{\geqq 0}(u_1 + u_3 + 2u_4) + \mathbb{R}_{\geqq 0}(u_2 + u_3 + 2u_4) + \mathbb{R}_{\geqq 0}(u_1 + u_2 + 2u_3 + 2u_4) + \mathbb{R}_{\geqq 0}(u_1 + 2u_2 + u_3 + 2u_4) + \mathbb{R}_{\geqq 0}(2u_1 + u_2 + u_3 + 2u_4)$. Then (Y, y) is 2-Gorenstein and $v_0 = \frac{1}{2}v_4$ satisfies the condition of Proposition 1.2. We see that $\sigma^* \cap M$ is generated by $\ell_1 = v_1 - v_2 - v_3 + v_4$, $\ell_2 = v_1 - v_2 + v_3$, $\ell_3 = v_1 + v_2 + v_3 - v_4$, $\ell_4 = v_1 + v_2 - v_3$, $\ell_5 = -v_1 - v_2 - v_3 + 2v_4$, $\ell_6 = -v_1 - v_2 + v_3 + v_4$, $\ell_7 = -v_1 + v_2 + v_3$, $\ell_8 = -v_1 + v_2 - v_3 + v_4$, $\ell_9 = v_1$, $\ell_{10} = -v_2 + v_4$, $\ell_{11} = v_3$, $\ell_{12} = v_2$, $\ell_{13} = -v_3 + v_4$ and $\ell_{14} = -v_1 + v_4$. (See Figure 3.) Let $z_i = e(\ell_i)$, for $i = 1$ through 14 and let $f = \sum_{1 \leqq i \leqq 14} z_i$. Then f is non-degenerate, (X, x) is an isolated singularity and $X \cap \mathrm{Sing}(Y) = \{x\}$. Let $\Sigma = \{$ faces of $\mathbb{R}_{\geqq 0}(u_1 + u_2 + u_3 + 2u_4) + \tau \mid \tau$ are 3-dimensional faces of $\sigma \}$ and let $\tilde{Y} = T_N\mathrm{emb}(\Sigma)$. Then $\Sigma = \Gamma^*(f)$ and $\tilde{Y}$ is the blowing up of Y along $y = \mathrm{orb}(\sigma)$. Although $\tilde{Y}$ has singularities, $\tilde{X} \cap \mathrm{Sing}(\tilde{Y}) = \phi$, where $\tilde{X}$ is the proper transformation of X under the blowing up $\Pi : \tilde{Y} \to Y$. Moreover, $\Pi^{-1}(y) \cap \tilde{X} = E(u_1+u_2+u_3+2u_4)$ is an Enriques surface. Each of small deformations $X_\varepsilon = \{ f = \varepsilon \}$ of X has eight isolated quotient singularities.

Figure 1

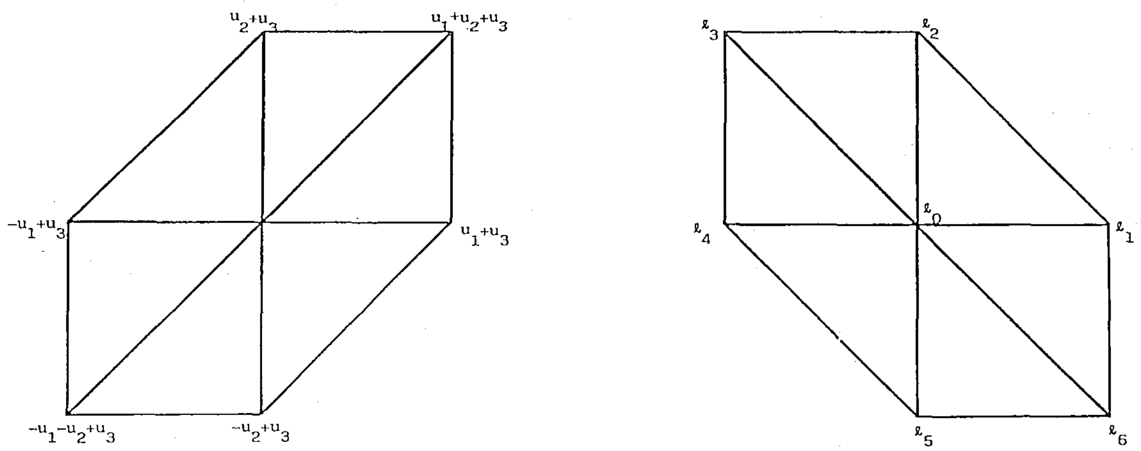


Figure 2

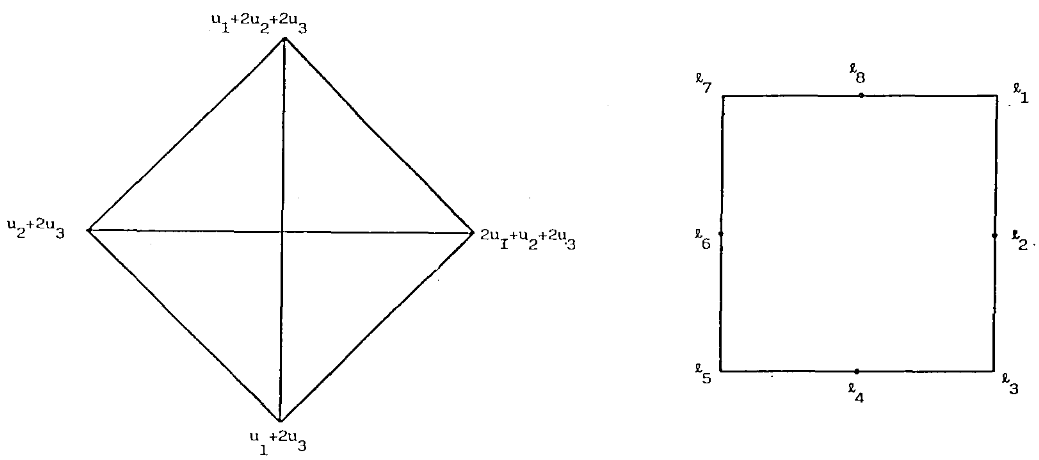
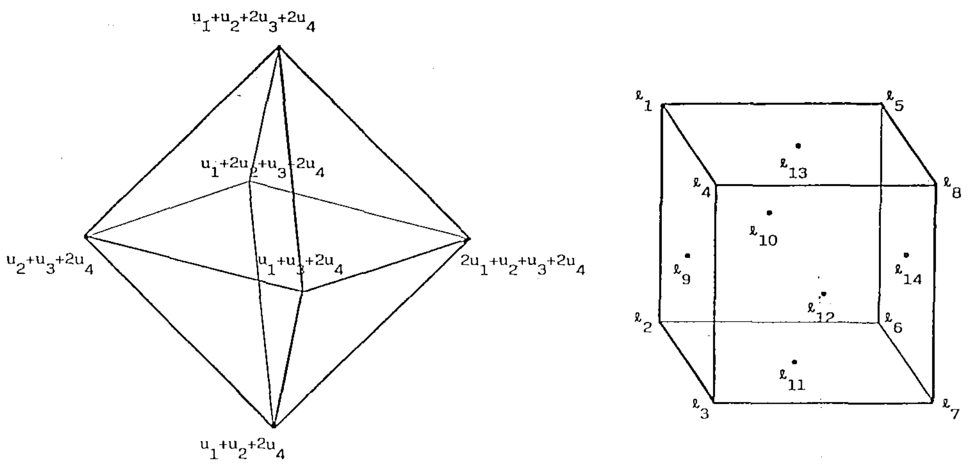


Figure 3



§3 Hyperplane sections of Gorenstein toric singularities

We keep the notations of the previous section and throughout this section, we assume that (Y, y) is an isolated (, i.e., each n-dimensional face of σ is non-singular), non-terminal and Gorenstein singularity. Hence (Y, y) is a canonical singularity of index 1 and the set $L := \{ u \in \mathrm{Int}(\sigma) \cap N \mid \langle v_0, u\rangle = 1 \}$ is non-empty, by [6, p294]. Moreover, we assume that $X = \{ f = 0 \}$ is a generic hyperplane section, i.e., $f = \sum c_v e(v)$ with $c_v \neq 0$, for the generators v of $\sigma^* \cap M$.

Proposition 3.1. Under the above assumptions, (X, x) is a purely elliptic singularity, i.e., $\delta_m(X, x) = 1$ for each positive integer m.

Proof. Let u_0 be an element in L. Then $\langle v_0, u_0\rangle = 1$ and $\{ v \in \sigma^* \mid \langle v, u_0\rangle \geqq 1 \} \supset$ the convex hull of $(\sigma^* \setminus \{0\}) \cap M = \Gamma_+(f) \ni v_0$. Hence the set $\{ v \in \sigma^* \mid \langle v, u_0\rangle = 1 \} \cap \Gamma_+(f)$ is a compact face of $\Gamma_+(f)$ and contains v_0. Therefore, $\delta_m = 1$ for each positive integer m, by Theorem 2.2. q.e.d.

Remark. If (Y, y) is non-terminal and canonical of index $r > 1$, then $v_0 \in \sigma^* \setminus \mathrm{Int}(\Gamma_+(f))$. There are examples with $v_0 \notin \Gamma_+(f)$, as well as examples with $v_0 \in \Gamma(f)$. Hence (X, x) may not be a periodically elliptic singularity in contrast with the above proposition.

Ishii[3] and Koyama independently showed that a 2-dimensional purely elliptic singularity is a simple elliptic singularity or a cusp singularity.

Proposition 3.2. When $n = 2$, (X, x) is a simple elliptic singularity (resp. a cusp singularity), if the cardinal number of L is equal to (resp. greater than) 1.

Proof. First, we consider the case that L consists of one element u_0. For each 2-dimensional face $\tau = \mathbb{R}_{\geqq 0}u' + \mathbb{R}_{\geqq 0}u''$ ($\{u', u''\} \subset \{u_1, u_2, \ldots, u_s\}$) of σ, $\{u_0, u', u''\}$ is a basis of N, because $\langle v_0, u_0\rangle = \langle v_0, u'\rangle = \langle v_0, u''\rangle = 1$ and the triangle spanned by u_0, u' and u'' contains no elements in N except u_0, u' and u''. Let $\{v_\tau, v', v''\}$ be the basis of M dual to $\{u_0, u', u''\}$. Then $\langle v_\tau, u_0\rangle = 1$ and $\langle v_\tau, u'\rangle = \langle v_\tau, u''\rangle = 0$. Hence σ^* is generated by v_τ and $\Gamma(f)$ consists of one face which is the polygon spanned by v_τ, for all 2-dimensional faces τ of σ. Therefore, $\Gamma^*(f) = \{$ faces of $\mathbb{R}_{\geqq 0}u_0 + \tau \mid \tau$ are 2-dimensional faces of σ $\}$ and the exceptional set $E = E(u_0)$ of the resolution of (X, x) obtained from $\Gamma^*(f)$ is a non-singular curve. It should be elliptic, because $\delta_m(X, x) = 1$.

When the cardinal number of L is greater than 1, we easily see that there exist at least two 2-dimensional compact faces of $\Gamma_+(f)$ containing v_0, which we denote by Δ_1 and Δ_2. Then $\Delta_1^* = \mathbb{R}_{\geqq 0}u_1'$ and $\Delta_2^* = \mathbb{R}_{\geqq 0}u_2'$ for primitive elements u_1' and u_2' in $\mathrm{Int}(\sigma) \cap N$ such that $\langle v_0, u_1' \rangle = d(u_1')$ and that $\langle v_0, u_2' \rangle = d(u_2')$. Hence the exceptional set E of the resolution $\pi : (\tilde{X}, E) \to (X, x)$ of (X, x) obtained from any subdivision of $\Gamma^*(f)$ contains two irreducible components $E(u_1')$ and $E(u_2')$ along which $\pi^*\omega$ has poles of order 1, by Lemma 2.3, where $\omega = \mathrm{Res}(e(v_0)((dw_1/w_1) \wedge \ldots \wedge (dw_{n+1}/w_{n+1}))/f)$. Therefore, (X, x) should not be a simple elliptic singularity. q.e.d.

Since (Y, y) is an isolated singularity, (X, x) is smoothable. On the other hand, Wahl [9, 10] showed that if a simple elliptic singularity (resp. a cusp singularity) (X, x) is smoothable, then $m(X) \leqq 9$, (resp. $m(X) - \ell(X) \leqq 9$), where $\ell(X)$ is the number of the irreducible components of the exceptional set E of the minimal resolution of (X, x) and $m(X)$ is the multiplicity of (X, x), which is equal to $-E^2$, if $-E^2 \geqq 3$.

Proposition 3.3. Assume that the cardinal number of L is equal to 1. If σ is an s-gonal cone, then $-E^2 = 12 - s$. (Therefore, $-E^2 \leqq 9$.)

Proof. Let $L = \{u_0\}$. Then $\Gamma^*(f) = \{$ faces of $\mathbb{R}_{\geqq 0}u_0 + \tau \mid \tau$ are 2-dimensional faces of σ $\}$ consists of non-singular cones, by the proof of Proposition 3.2. Hence we obtain resolutions $\Pi : (\tilde{Y}, F) \to (Y, y)$ and $\pi = \Pi_{|\tilde{X}} : (\tilde{X}, E) \to (X, x)$, where $\tilde{Y} = T_N\mathrm{emb}(\Gamma^*(f))$, F is the closure of $\mathrm{orb}(\mathbb{R}_{\geqq 0}u_0)$, $\tilde{X}$ is the proper transformation of X under Π and $E = \tilde{X}\cdot F$. Let $\tilde{D}_i$ be the proper transformation of D_i under Π and let $E_i = F\cdot\tilde{D}_i$. Since $F + \tilde{X} = [\Pi^*f]$ and $F + \tilde{D}_1 + \tilde{D}_2 + \ldots + \tilde{D}_s = [\Pi^*e(v_0)]$ are principal divisors, we have $-E^2_{|\tilde{X}} = -F^2\cdot\tilde{X} = F\cdot\tilde{D}^2 = \sum_{1\leqq i\leqq s}F\cdot\tilde{D}_i^2 + 2\sum_{1\leqq i<j\leqq s}F\cdot\tilde{D}_i\cdot\tilde{D}_j = (\sum_{1\leqq i\leqq s}E^2_{i|F}) + 2s = 3(4 - s) + 2s = 12 - s$, because F is a non-singular toric variety whose 1-dimensional orbits are E_1, E_2, ... and E_s. q.e.d.

Proposition 3.4. Assume that the convex hull of L is a polygon. If σ is an s-gonal cone, then $-E^2 - \ell(X) = 12 - s$. (Therefore, $-E^2 - \ell(X) \leqq 9$.)

Proof. Let P (resp. Q) be the convex hull of L (resp. $\{ u \in \sigma \cap N \mid \langle v_0, u\rangle = 1 \}$). Then $Q = \{ u \in \sigma \mid \langle v_0, u\rangle = 1 \}$ and $\mathrm{Int}(Q) \supset P$. Take a triangulation Δ (resp. Δ') of P (resp. $Q \setminus \mathrm{Int}(P)$) so that the set of the vertices of Δ (resp. Δ') agrees with $P \cap N = L$ (resp. $(Q \setminus \mathrm{Int}(P)) \cap N$). Let e_0, e_1 and e_2 (resp. e_0', e_1' and e_2') be the numbers of the vertices, edges and faces, respectively, of Δ (

resp. Δ'). Then $e_0 - e_1 + e_2 = 1$ and $e'_0 - e'_1 + e'_2 = 0$, because P and Q are polygons. Let ℓ be the number of the vertices on the boundary ∂P of P. Then $e'_0 = \ell + s$ and $3e'_2 = 2e'_1 - (\ell + s)$, because the number of the vertices (resp. edges) on the boundary of $Q \setminus \mathrm{Int}(P)$ is equal to $\ell + s$. Hence by an easy calculation, we have $e'_1 = 2(\ell + s)$. Since $\square := \Delta \cup \Delta'$ is a triangulation of Q, we see that $\Sigma^* := \{ \mathbb{R}_{\geq 0}\tau \mid \tau$ are simplexes of $\square \} \cup \{0\}$ is a subdivision of $\Gamma^*(f)$ and consists of non-singular cones. Hence we have a resolution $\Pi : (\tilde{Y}, F) \to (Y, y)$, where $\tilde{Y} = T_N\mathrm{emb}(\Sigma^*)$. Let $\tilde{D}_i$ be the proper transformation of D_i under Π and let $\tilde{D} = \tilde{D}_1 + \tilde{D}_2 + \ldots + \tilde{D}_s$. Then Δ and $\square$ are the dual graphs of $F = F_1 + F_2 + \ldots + F_{e_0}$ and $F + \tilde{D}$, respectively. Since $\tilde{X} + F = [\Pi^* f]$ and $\tilde{D} + F = [\Pi^* e(v_0)]$ are principal divisors, we have $0 = F_i \cdot F_j \cdot (\tilde{D} + F) = F_i^2 \cdot F_j + F_i \cdot F_j^2 + 2$, if $F_i \cap F_j \neq \emptyset$ and $-E^2_{|\tilde{X}} = -F^2 \cdot \tilde{X} = F \cdot \tilde{D}^2 = \sum_{1 \leq i \leq e_0} (\sum_{1 \leq j \leq s} F_i \cdot \tilde{D}_j^2 + 2\sum_{1 \leq j < k \leq s} F_i \cdot \tilde{D}_j \cdot \tilde{D}_k) = \sum_{1 \leq i \leq e_0, 1 \leq j \leq s} F_i \cdot \tilde{D}_j^2 + 2s$, where $\tilde{X}$ is the proper transformation of X under Π and $E = \tilde{X} \cdot F$. On the other hand, since each irreducible component F_i of F is a non-singular toric variety with $F_i \setminus (F + \tilde{D} - F_i)$ as the algebraic torus, we have $\sum_{i \neq j} F_i \cdot F_j^2 + \sum_{1 \leq k \leq s} F_i \cdot \tilde{D}_k^2 = 3(4 - d_i)$, where d_i is the number of the double curves on F_i. Hence by taking the sum of the self-intersection numbers of the double curves $F_i \cdot F_j$ and $F_i \cdot \tilde{D}_k$ on the irreducible components F_i of F , we have $-2e_1 + \sum_{1 \leq i \leq e_0, 1 \leq j \leq s} F_i \cdot \tilde{D}_j^2 = \sum_{1 \leq i \leq e_0} 3(4 - d_i) =$

$12e_0 - 3(2e_1 + \ell + s) = 12e_0 - 6e_1 - 3\ell - 3s$. Therefore, $-E^2_{|\tilde{X}} = 12e_0 - 6e_1 - 3\ell - 3s + 2e_1 + 2s = 12e_0 - 4e_1 - 3\ell - s = 12e_0 - 12e_1 + 12e_2 + \ell - s = 12 + \ell - s$, because $3e_2 = 2e_1 - \ell$. Thus we obtain $-E^2 - \ell = 12 - s$. Here we note that ℓ is equal to the number of the irreducible components of E, because $\tilde{X} \cap F_i \neq \emptyset$, if and only if $\tilde{D} \cap F_i \neq \emptyset$ and then $\tilde{X} \cap F_i$ is irreducible. Also we note that although $(\tilde{X}, E)$ is not a minimal resolution, the contraction of a rational curve E_i with $E_i^2 = -1$ does not change the number $-E^2 - \ell$, because E is a cycle of curves. Thus we complete the proof.

q.e.d.

Examples. In the following table, $E = E_1 + E_2 + \ldots + E_\ell$ is the exceptional set of the minimal resolution of (X, x) such that $E_i \cdot E_{i+1} = 1$ for each $i \in \mathbb{Z}/\ell\mathbb{Z}$.

generators of σ	ℓ	$-E_1^2, -E_2^2, \ldots, -E_\ell^2$
(0, 0, 1), (5, 2, 1), (3, 5, 1)	6	5, 4, 5, 4, 5, 4
(0, 0, 1), (4, 1, 1), (3, 4, 1)	6	7, 2, 7, 2, 7, 2
(0, 0, 1), (8, 3, 1), (5, 8, 1)	9	5, 4, 3, 5, 4, 3, 5, 4, 3
(0, 0, 1), (7, 2, 1), (5, 7, 1)	9	5, 5, 2, 5, 5, 2, 5, 5, 2
(0, 0, 1), (7, 3, 1), (4, 7, 1)	9	6, 4, 2, 6, 4, 2, 6, 4, 2

§4 Deformations

We assume that $n = \dim X \geqq 3$, throughout this section. Let $U = X \setminus \{x\}$ and let $W = Y \setminus \{y\}$. Then we have the isomorphism $T_X^1 \simeq H^1(U, \Theta_U)$, by Proposition 2.1 and [7, Theorem 2], where $T_X^1 = H^0(X, \Theta_X^1)$ is the tangent space to the formal moduli space of X and Θ_U is the tangent sheaf of U. Consider the long exact sequence arising from the short exact sequence of sheaves:

$$0 \to \Theta_U \to i^*\Theta_W \to \mathcal{N} \to 0,$$

where $i : U \hookrightarrow W$ is the inclusion map. Here we note that the normal sheaf $\mathcal{N} \simeq \mathcal{O}_U(U)$ is isomorphic to the structure sheaf $\mathcal{O}_U$, because X is a principal divisor on Y. Let $\{\theta_1, \theta_2, \ldots, \theta_\ell\}$ be a basis of the image of the map $\delta : H^0(U, \mathcal{N}) \to H^1(U, \Theta_U)$ and let g_i be an element of $H^0(Y, \mathcal{O}_Y)$ whose image is θ_i under the composite of the surjective maps $H^0(Y, \mathcal{O}_Y) = H^0(W, \mathcal{O}_W) \to H^0(U, \mathcal{O}_U) \simeq H^0(U, \mathcal{N})$ and $H^0(U, \mathcal{N}) \to \mathrm{Im}(\delta)$. Let $\mathfrak{X} = \{ (z, t) \in Y \times \Delta \mid f(z) + t_1 g_1(z) + t_2 g_2(z) + \ldots + t_\ell g_\ell(z) = 0 \}$ and let π be the restriction to $\mathfrak{X}$ of the projection $Y \times \Delta \to \Delta$, where $\Delta = \{ (t_1, t_2, \ldots, t_\ell) \in \mathbb{C}^\ell \mid |t_j| < \varepsilon \}$. Then π is flat, by [1, Chapter V, Corollary 1.5]. Let $\mathfrak{U}$ be the open set of $\mathfrak{X}$ on which π is smooth. Then we obtain a family $\pi_{|\mathfrak{U}} : \mathfrak{U} \to \Delta$ of deformations of the complex manifold U. Moreover, by an easy calculation, we have $\rho(\frac{\partial}{\partial t_j}) = \theta_j$

for $j = 1$ through ℓ, where $\rho : T_0(\Delta) \to H^1(U, \Theta_U)$ is the infinitesimal deformation map. Hence ρ is injective and if $H^1(U, i^*\Theta_W) = 0$, then ρ is surjective.

Theorem 4.1. If $H^1(U, i^*\Theta_W) = 0$, then $\pi : \mathfrak{X} \to \Delta$ is a locally semiuniversal family of X.

Proof. Recall that T_X^1 is defined by the exact sequence

$$0 \to Hom(\Omega_X^1, \mathcal{O}_X) \to Hom(j^*\Omega_{\mathbb{C}^N}^1, \mathcal{O}_X) \to Hom(I/I^2, \mathcal{O}_X) \to T_X^1 \to 0$$

obtained by the exact sequence of sheaves: $I/I^2 \xrightarrow{d} j^*\Omega_{\mathbb{C}^N}^1 \to \Omega_X^1 \to 0$, for an inclusion $j : X \hookrightarrow \mathbb{C}^N$ with the ideal sheaf I. On the other hand, we have the exact sequence

$$\begin{aligned} 0 &\to Hom(\Omega_X^1, \mathcal{O}_X) \to Hom(j^*\Omega_{\mathbb{C}^N}^1, \mathcal{O}_X) \to Hom(Im(d), \mathcal{O}_X) \\ &\to Ext_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) \to 0, \end{aligned}$$

by the short exact sequence of sheaves: $0 \to Im(d) \to j^*\Omega_{\mathbb{C}^N}^1 \to \Omega_X^1 \to 0$. Since the support of $ker(d)$ is $\{x\}$, we have $Hom(Im(d), \mathcal{O}_X) = Hom(I/I^2, \mathcal{O}_X)$. Thus we have the canonical isomorphism $Ext_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X) \simeq T_X^1$. Hence the infinitesimal deformation map $T_0(\Delta) \to Ext_{\mathcal{O}_X}^1(\Omega_X^1, \mathcal{O}_X)$ for the family $\pi : \mathfrak{X} \to \Delta$ is bijective. Then by [8, Theorem 6.1], $\pi : \mathfrak{X} \to \Delta$ is locally semiuniversal. q.e.d.

Corollary 4.2. If $H^1(U, i^*\Theta_W) = 0$, then any small deformation of X is also a hypersurface section of Y.

Proposition 4.3. If σ is a simplicial cone (hence Y is a quotient of $\mathbb{C}^{n+1}$), then $H^1(U, i^*\Theta_W) = 0$.

Proof. Let ℓ_1, ℓ_2, ... and ℓ_{n+1} be the generators of σ and let $N' = \mathbb{Z}\ell_1 + \mathbb{Z}\ell_2 + \ldots + \mathbb{Z}\ell_{n+1}$. Here we may assume that ℓ_1, ℓ_2, ... and ℓ_{n+1} are primitive elements in N. Then the inclusion $N' \hookrightarrow N$ induces a holomorphic map $\varphi : Y' \to Y$, where $Y' = T_{N'}\mathrm{emb}(\{$ faces of $\sigma\}) \simeq \mathbb{C}^{n+1}$. Let $U' = \varphi^{-1}(U)$. Then $\varphi_{|U'} : U' \to U$ is unramified, by the assumption $X \cap \mathrm{Sing}(Y) = \{x\}$. Hence $H^1(U, i^*\Theta_W) = H^1(U', h^*\Theta_{Y'})^G = 0$, where $h : U' \hookrightarrow Y'$ is the inclusion map and G is the covering transformation group of φ. q.e.d.

Example. Let X' be the hypersurface of $\mathbb{C}^4$ defined by $z_1^2 + z_2^6 + z_3^6 + z_4^6 = 0$ and let $X = X'/G$ be the quotient of X' under the group G generated by $(1, \xi, \xi, \xi)$, where ξ is a primitive cube root of 1. Then X is a hypersurface section of $Y = \mathbb{C}^4/G$, which is a toric singularity, and whose singular locus $\mathrm{Sing}(Y)$ is 1-dimensional. We easily see that X has an isolated singularity obtained by contracting a K3 surface. By Corollary 4.2 and Proposition 4.3, any small deformation of X is also a hypersurface section $X_t = \pi^{-1}(t)$ of Y. Since X_t

intersect Sing(Y) at finitely many points, X_t has singularities, i.e., X is not smoothable.

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On Right Equivalence of Map-germs

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The study of $\mathcal{G}$-equivalence of map-germs is the basis of singularity theory. In the cases $G = \mathscr{C}$, $\mathscr{K}$, $\mathscr{L}$ and $\mathscr{A}$, a number of interesting criteria for $\mathcal{G}$-equivalence of germs are known (cf. [M1], [M2], [B-G-P] and [G]). $\mathscr{R}$-equivalence of map-germs has been studied by many mathematicians for more than twenty years. And it seems to still be concerned with this problem till now.

1. $\mathscr{R}$-equivalence of Function-germs

1.1 The problem of right equivalence for function-germs can be reduced to a problem of contact equivalence in symplectic geometry. In 1968, Tougeron had already described in symplectic form a theorem about right equivalence (see [To, p.209] or [G-G, p.384]). Using this result, Golubitsky and Guillemin get the following

Proposition ([G-G]). Suppose that C^∞ function-germs $f, g: (N, x_0) \longrightarrow (R, 0)$ satisfy the Milnor condition (i.e. they have finite $\mathscr{R}$-codimension) with $df(x_0) = dg(x_0) = 0$. Then f and g are $\mathscr{R}$-equivalence iff

(1) There is an isomorphism $\gamma: C_N/J(f)^2 \longrightarrow C_N/J(g)^2$ s.t. $\gamma(\bar{f}) = \bar{g}$, where $J(f)$ and $J(g)$ denote the Jacobian ideals of f and g in C_N, and $\bar{f}$, $\bar{g}$ are the projections of f, g respectively.

(2) The rank and signature of the Hessians $H(f)$ and $H(g)$ are equal.

Remark 1. The condition that f and g have finite $\mathscr{R}$-codimension is stronger, because it means that f and g are all finitely $\mathscr{R}$-determined.

2. du Plessis and Wilson pointed out that the statement also holds true in the real-analytic or complex-analytic cases and the condition (2) can be

deleted in complex-analytic case. And they give another proof of it as one of their simpler applications (see [P-W, p.171]).

1.2 By virtue of basic idea of the sufficiency part of Mather's theorem on finitely right-determined map-germs, von Grudzinski found sufficient conditions for two function-germs to be right equivalent (see [Gr]).

First let us introduce some notations. Let $\mathscr{E}(n,m)$ denote the set of all C^∞ map-germs $(R^n, 0) \longrightarrow R^m$ and $L(n)$ the group of all invertible C^∞ map-germs $(R^n, 0) \longrightarrow (R^n, 0)$. $\mathscr{E}(n) = \mathscr{E}(n, 1)$. It has a maximal ideal $\mathscr{M}(n)$ consisting of all germs $(R^n, 0) \longrightarrow (R, 0)$. For an arbitrary subset I of $\mathscr{E}(n)$, $L_I(n)$ is the subgroup of $L(n)$ generated by $\{ \phi \in L(n): (\phi - id_{R^n})_i \in I$ for every $1 \leq i \leq n \}$. Now we state some results of Grudzinski.

Theorem. Let $f, g \in \mathscr{E}(n)$, and let I be a finitely generated proper ideal in $\mathscr{E}(n)$ such that

$$g - f \in I \cdot J(f) \quad \text{and} \quad I \cdot J(g - f) \subset \mathscr{M}(n) I \cdot J(f).$$

Then f and g are right equivalent by an equivalence belonging to $L_I(n)$.

Corollary. Let $r, p \in N$, $f \in \mathscr{M}(n)^p$ and $g \in \mathscr{E}(n)$ such that

$$g - f \in \mathscr{M}(n)^{r+1} J(f) \text{ and } (g - f)\mathscr{M}(n)^{r+1} \subsetneq \mathscr{M}(n)^{r+4-p} J(f)^2.$$

Then f and g are right equivalent by an equivalence $\phi \in L(n)$ such that $j^r(\phi - id_{R^n}) = 0$.

Remark. The corollary contains the following result of Thom as its special case (see [T, p.60]):

Let $f, g \in \mathscr{E}(n)$. If there is $1 \leq p \leq 3$ such that $f \in \mathscr{M}(n)^p$ and $g - f \in \mathscr{M}(n)^{3-p} J(f)^2$, then f and g are right equivalent (by an equivalence $\phi \in L(n)$ satisfying $j^1(\phi - id_{R^n}) = 0$).

1.3 Associated with deformation theory, I obtained the following

Proposition. ([L, p.6]). Let f, g: $(R^p \times R^q, 0) \longrightarrow (R, 0)$ be p-parameter universal deformations of f_0 and g_0, where $f_0 = f|\{0\} \times R^q$ and $g_0 = g|\{0\} \times R^q$, and p=codim f_0 = codim g_0. If f_0 and g_0 are right equivalent, so are f and g.

1.4 S. S.-T, Yau had already characterised right equivalence of holomorphic function-germs. We will describe his good result as follows.

Let Θ_{n+1} denote the ring of all holomorphic function-germs $(C^{n+1}, 0) \longrightarrow (C, 0)$. For given $f \in \Theta_{n+1}$, we set

$$Q(f) = \Theta_{n+1}/\Lambda(f)$$

where $\Lambda(f)$ is the ideal of f in Θ_{n+1} generated by all first partial derivatives of f. Q(f) has a natural $C\{t\}/(t^{n+1})$ algebra structure.

Theorem ([Y, p.292]). Let f, g: $(C^{n+1}, 0) \longrightarrow (C, 0)$ be germs at the origin of holomorphic functions. Then f is right equivalent to g iff Q(f) is isomorphic to Q(g) as a $C\{t\}/(t^{n+1})$ algebra.

2. Right Equivalence of Map-germs

We will restrict our attention to cases where source dimension $\geq$ target dimension. First we discuss necessary conditions for two map-germs to be $\mathscr{R}$-equivalent.

2.1 Let f: $(N,x_0) \longrightarrow (M,y_0)$ be a C^∞ map-germ. Suppose that $\mathscr{E}^*_{(N,x_0)}$ denote the C_N-module of all germs of differential forms at x_0 in N. It is a graded algebra over R. We define δf: $\mathscr{E}^*_{(M,y_0)} \longrightarrow \mathscr{E}^*_{(N,x_0)}$ by

$$\delta f(\omega)|_x = \delta f_x(\omega|_{f(x)}), \quad \omega \in \mathscr{E}^*_{(M,y_0)}, \quad x \in (N,x_0)$$

where δf_x is the transposition of the differential df_x. δf is an algebra homomorphism.

Let π_1 and π_2 denote the natural projections of $(N\times M,(x_0,y_0))$ on (N,x_0) and (M,y_0) respectively. Let $\{\omega_i | i=1,\cdots,m\}$ be a basis for all germs of differential 1-forms on (M,y_0).

Proposition ([L, p.2]). Suppose that f, g: $(N,x_0) \to (M,y_0)$ are $\mathscr{R}$-equivalent. Then

(i) $\delta f(\mathscr{E}^*_{(M,y_0)})$ and $\delta g(\mathscr{E}^*_{(M,y_0)})$ considered as subalgebras in $\mathscr{E}^*_{(N,x_0)}$ are equivalent.

(ii) graph(f) and graph(g) considered as subset-germs in $(N\times M, (x_0,y_0))$ are integral manifold germs of ideals Δ_f and Δ_g in $\mathscr{E}^*_{(N\times M,(x_0,y_0))}$, which are generated by

$$\{\delta\pi_1 \delta f(\omega_i) - \delta\pi_2(\omega_i) \mid i=1,\cdots,m\}$$

and

$$\{\delta\pi_1 \delta g(\omega_i) - \delta\pi_2(\omega_i) \mid i=1,\cdots,m\}$$

respectively,

(iii) Moreover, graph(f) and graph(g) satisfy the following relation

$$\mathrm{graph}(f) = (h\times 1_M)(\mathrm{graph}(g))$$

where h: $(N,x_0)\longrightarrow(N,x_0)$ is an invertible C^∞ map-germ such that $f=g\circ h^{-1}$, and 1_M denotes the identity map-germ on (M,y_0).

2.2 The study of $\mathscr{R}$-equivalence of map-germs is related to that of Thom-Boardman singularities. We will show it below.

First we define $j: \mathscr{E}(n)\longrightarrow\mathscr{E}(n)/\mathscr{M}(n)^\infty$ as follows: if $\alpha\in\mathscr{E}(n)$, then $j(\alpha)=\hat{\alpha}$ is the ∞-jet of α at the origin. According to Borel's theorem, $\mathscr{E}(n)/\mathscr{M}(n)^\infty = R[[x_1,\cdots,x_n]]$ (abbreviated as A_n), which is the formal power series algebra on n coordinate functions $x_1,\cdots,x_n$ over R. Now we define two operators δ and β on the set of proper ideals in A_n. Let $\mathscr{T}$ be such an ideal. Set

(i) $\delta\mathscr{T} = \Delta_{r+1}\mathscr{T}$, which is a Jacobian extension of $\mathscr{T}$ and r = rank $\mathscr{T}$,

(ii) $\beta\mathscr{T} = \mathscr{T} + (\delta\mathscr{T})^2 + (\delta^2\mathscr{T})^3 + \cdots + (\delta^{k-1}\mathscr{T})^k + \cdots$.

(see [M3, p.236]).

For any $f \in \mathscr{E}(n,m)$, f can be indicated as $f = (f_1,\cdots,f_m)$, where $f_i \in \mathscr{E}(n)$ is the ith component of f for $i=1,\cdots,m$. Hence we define $\mathscr{T}(f)$ to be the ideal in A_n generated by $\widehat{f}_1,\cdots,\widehat{f}_m$.

By the Boardman symbol of f, we mean the sequence of integers $I(f) = (i_1,\cdots,i_k,\cdots)$, where $i_1=\operatorname{corank}\mathscr{T}(f)$, $i_2=\operatorname{corank}\delta\mathscr{T}(f),\cdots$, $i_k=\operatorname{corank}\delta^{k-1}\mathscr{T}(f),\cdots$.

Proposition ([L, p.4]). Let f, $g \in \mathscr{E}(n,m)$. If f and g are right equivalent, then

(i) $I(f) = I(g)$ and

(ii) $\beta(\mathscr{T}(f))$ is equivalent to $\beta(\mathscr{T}(g))$.

Remark. In [L] there is an example showing that the inverse of the proposition is false.

2.3 For the convenience of discussion below, we introduce the concept of $\mathscr{R}_J$-equivalence.

Let f: $(N,x_0)\longrightarrow(P,y_0)$ be a C^τ map-germ, here τ may be any of $C-\omega$ ("complex-analytic"), $R-\omega$ ("real-analytic") or ∞. Let $S\subset C_N^{\times p}$ be a C_N-submodule and $\{y_1,\cdots,y_p\}$ a choice of coordinates for (P,y_0).

Definition. A C^τ map-germ g: $(N,x_0)\longrightarrow(P,y_0)$ is an S-approximation to f (w.r.t. $\{y_1,\cdots,y_p\}$) if

$$(y_1\circ f-y_1\circ g,\cdots,y_p\circ f-y_p\circ g)\in S$$

(abbreviated as $f-g\in S$ (w.r.t. $\{y_1,\cdots,y_p\}$)).

Remark. (1) Since Θ_f can be identified with $C_N^{\times p}$, we may have the notion of S'-approximation to f, where $S'\subset\Theta_f$ is a C_N-submodule.

(2) In particular, if $S = I^{\times p}$ (or $S' = I\Theta_f$) for some ideal $I\subset C_N$, then we will write I-approximation instead of S-approximation.

Definition (i) Let $J\subset C_N$ be an ideal. $\mathscr{R}_J$ is the subgroup of $\mathscr{R}$ consisting of diffeomorphisms of (N,x_0) which are J-approximations to the identity.

(ii) Let f, g: $(N,x_0)\longrightarrow(P,y_0)$ be map-germs. We say f is $\mathscr{R}_J$-equivalent to g if there is $h\in\mathscr{R}_J$ such that $f=g\circ h^{-1}$.

Now we describe a necessary condition on $\mathscr{R}_J$-equivalence which is due to du Plessis and Wilson.

Proposition ([P-W, p.165]). Let f: $(N,x_0)\longrightarrow(P,y_0)$ be a C^r map-germ. Let $J\subset J(f)$ be an ideal in C_N, where J(f) is the ideal generated by the determinants of the p×p minors of the matrix for df obtained via any choices of coordinates for (N,x_0), (P,y_0). If g: $(N,x_0)\longrightarrow(P,y_0)$ is $\mathscr{R}_J$-equivalent to g, then

$$f-g\in J\cdot tf(\Theta_N).$$

2.4 There are several sufficient conditions for right equivalence of map-germs which are significant.

Proposition ([Wi, p.238]). Let f_0, f_1: $(C^n,0)\longrightarrow(C^p,0)$ be holomorphic map-germs such that

(i) $\Sigma(f_0)=\Sigma(f_1)=C$, where $\Sigma(f_0)$ and $\Sigma(f_1)$ denote the critical sets of f_0 and f_1 in C^n respectively,

(ii) $f_0|C=f_1|C$,

(iii) $Tf_0|C=Tf_1|C$,

(iv) $f_1-f_0\in J(f_0)^2\cdot\mathscr{M}\cdot\Theta_n^{\times p}$ if $\operatorname{rank} f_0=p-1$.

Then there is an automorphism h of $(C^n,0)$ such that $f_1\circ h=f_0$.

C. T. C. Wall proved the following splitting theorem using the result above.

Theorem. Let f_i: $(E^s,0)\longrightarrow(E^2,0)$, g_i: $(E^t,0)\longrightarrow(E^2,0)$ ($i=1,2$; E= R or C) be map-germs of cokernel rank 2 whose 2-jets define non-degenerate pencils of quadratic forms. Let e_{f_i}, (respectively e_{g_i})(i=1, 2), be the set of eigenvalues of $j^2 f_i$ (respectively $j^2 g_i$).

Suppose $e_{f_1} = e_{f_2}$ and $e_{g_1} = e_{g_2}$, but $e_{f_i} \cap e_{g_i} = \emptyset$.

Let $F_i = f_i \oplus g_i$: $E^s \oplus E^t \longrightarrow E^2$. Suppose that f_1, g_1 have non-degenerate critical sets.

Then F_1 and F_2 are $\mathcal{R}$-equivalent if and only if f_1, f_2 are $\mathcal{R}$-equivalent and g_1, g_2 are $\mathcal{R}$-equivalent.

Remark. The argument is due to Wall in the C-analytic case, and the proof is given by du Plessis and Wilson in the C-analytic, R-analytic and C^∞ cases. One may refer to [W, p.447] (for the former) and [P-W, p.172] (for the later).

2.5 Finally we state two deeper results in [P-W].

Theorem. Let f, g: $(N, x_0) \longrightarrow (P, y_0)$ be map-germs. Let $J \subset J(f)$ be an ideal in C_N. Suppose that

$$f - g \in J \cdot tf(\Theta_N)$$

(i) If $J \subset \mathcal{M}_N^2$, then f and g are $\mathcal{R}_J$-equivalent.

(ii) If $J \not\subset \mathcal{M}_N^2$:

A. Suppose that f is a submersion. Then f, g are $\mathcal{R}_J$-equivalent iff g is a submersion.

B. Suppose that f has cokernel rank one and non-zero full second intrinsic derivative. Then the following are equivalent:

(a) f, g are $\mathcal{R}_J$-equivalent,

(b) $j^2 f$, $j^2 g$ are $\mathcal{R}$-equivalent,

(c) J(f)=J(g) and, in the C^∞ and R-analytic cases, the Hessians H(f), H(g) have the same index,

(d) g has cokernel rank one and the Hessians H(f), H(g) have the same rank and, in the C^∞ and R-analytic cases, the same index.

The next theorem is the main result in [P-W].

Theorem. Let $f: (N,x_0)\longrightarrow(P,y_0)$ be a map-germ with non-degenerate critical set.

Let $g: (N,x_0)\longrightarrow(P,y_0)$ be a map-germ satisfying

(α) $\Sigma(f)\subset\Sigma(g)$

(β) $f|\Sigma(f) = g|\Sigma(g)$

(γ) $\mathrm{Im}Tg_x\subset \mathrm{Im}Tf_x$ for x in some τ-dense subset of $\Sigma(f)$.

(i) Suppose that f has cokernel rank > 1, or that f has cokernel rank 1 and zero full second intrinsic derivative. Then f, g are $\mathscr{R}_{J(f)}$-equivalent.

(ii) Suppose that f has cokernel rank 1 and non-zero full second intrinsic derivative. Then the following are equivalent:

(a) f, g are $\mathscr{R}_{J(f)}$-equivalent.

(b) j^2f, j^2g are $\mathscr{R}$-equivalent.

(c) $J(f) = J(g)$ and, in the C^∞ and R-analytic cases, the Hessians $H(f)$, $H(g)$ have the same index.

(c') $\Sigma(f) = \Sigma(g)$, g has non-degenerate critical set, and, in the C^∞ and R-analytic cases, the Hessians $H(f)$, $H(g)$ have the same index.

(d) g has cokernel rank 1 and the Hessians $H(f)$, $H(g)$ have the same rank, and, in the C^∞ and R-analytic cases, the same index.

Remark 1. That f has non-degenerate critical set means that

(1) $I(\Sigma(f)) = J(f)$, where $I(\Sigma(f))$ is the ideal of C^τ Function-germs on (N,x_0) vanishing on the subset-germ $(\Sigma(f),x_0)$,

(2) $\dim\Sigma(f) = p-1$.

2. If map-germs f, g are $\mathscr{R}_{I(\Sigma(f))}$-equivalent, then they must satisfy (α), (β) and (γ) (indeed with equality in (α), (γ)). In fact, we

have the following

Proposition ([P-G, p.168]). Let f, g: $(N,x_0)\longrightarrow(P,y_0)$ be map-germs. Suppose that f has non-degenerate critical set. Then

$$f - g \in J(f)\cdot tf(\Theta_N)$$

if and only if (α), (β) and (γ) hold true.

It is not easy to find the conditions for two map-germs to be right equivalent. How does one ought to characterise right equivalence in general? It seems to be an open problem.

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Wave propagation and the theory of singularities

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§1. Introduction.

The subject we will consider here is linear and nonlinear wave propagation. The equations which represent the phenomena of wave propagation are generally hyperbolic and the waves are mathematically translated as the singularities of solutions. The aim of this talk is to explain how the above problems may have connection with the theory of singularites, real and complex both. The subjects treated here are restricted to those which the author is now interested in. In §2, we will treat linear hyperbolic equations with constant coefficients. In §3, we will explain the passage from linear hyperbolic equations to nonlinear partial differential equations of first order. In the last section, we will sketch how to apply the theory of singularities of smooth mappings to construct the singularities of generalized solutions of first order partial differential equations.

§2. Linear hyperbolic equations with constant coefficients.

Let $P(D_x)$ be a hyperbolic partial differential operator of order m with respect to $\vartheta \in R^n/\{0\}$ where $x \in R^n$ and $D_x = -i(\partial/\partial x_1, \ldots, \partial/\partial x_n)$, i.e., its characteristic polynomial $P(\xi)$ satisfies the following two properties: i) $P_m(\vartheta) \neq 0$ where $P_m(\xi)$ is the principal part of $P(\xi)$, and ii) There exists a constant $\gamma_0 > 0$ such that $P(\xi - i\gamma\vartheta) \neq 0$ for all $\xi \in R^n$ and $\gamma \geqq \gamma_0 > 0$. The notion of hyperbolicity was formulated by L. Gårding [5].

The operator $P(D_x)$ has the elementary solution $E(P)(x)$

$$(2.1)\quad E(P)(x) = (2\pi)^{-n} \int_{R^n - i\gamma\vartheta} \frac{e^{i<x,\zeta>}}{P(\zeta)}\, d\zeta$$

whose support is contained in $\{x \in R^n;\ <x,\vartheta> \geqq 0\}$. For a distribution $u(x)$, Su is the support of u and SSu is the singular support of u which is the smallest and closed subset such that $u(x) \in C^\infty(R^n - SSu)$. The problem of this section is concerned with the complete description of $SS\, E(P)$ for any hyperbolic operator $P(D_x)$. This is one of the classical and well studied problems. Here we assume always that $P(D_x)$ is homogeneous differential operator. We put $A=\{\xi \in R^n/\{0\};\ P(\xi)=0\}$. Roughly speaking, one can say that the singularities of $E(P)$ are caused by the singularities of the kernel of the representation (2.1). In this case, the kernel has the pôles on the algebraic surface A. At first, we consider the case where A has no singular point, i.e., if $P(\xi^0)=0$ $(\xi^0 \neq 0)$, then $(\mathrm{grad}\, P)(\xi^0) \neq 0$. Let $\xi^0=(\xi^0_1, \ldots, \xi^0_n) \in A$. Then the first approximation of $P(\xi)$ at $\xi=\xi^0$ is given by

$$P_{\xi^0}(\xi) \underset{\mathrm{def}}{=} \Sigma_{j=1}^n \frac{\partial P}{\partial \xi_j}(\xi^0)\ (\xi_j - \xi_j^0) = \Sigma_{j=1}^n \frac{\partial P}{\partial \xi_j}(\xi^0)\ \xi_j\ .$$

If we replace $P(\zeta)$ by $P_{\xi^0}(\zeta)$ in (2.1), we can easily calculate it, i.e., we get the exact representation of $E(P_{\xi^0})(x)$. Next we sum up the support of $E(P_{\xi^0})$ with respect to ξ^0, then we get the classical and well known theorem as follows:

Theorem 1. Suppose that A has no singular point, then we get

$$SS\ E(P) = \bigcup_{\xi^0 \in A} \{x = k(\mathrm{grad}\ P)(\xi^0)\ ;\ k \in R^1\} \cup \{x;\ \langle x,\zeta\rangle \geqq 0\}.$$

Remark. $\bigcup_{\xi^0 \in A} \{x = k(\mathrm{grad}\ P)(\xi^0);\ k \in R^1\}$ is the real dual algebraic surface of A.

The problem is to treat the case where A has the singularities, i.e., there exists a point $\xi^0 \neq 0$ such that $P(\xi^0)=0$ and $(\mathrm{grad}\ P)(\xi^0)=0$. Concerning this subject, Atiyah-Bott-Gårding [1], [2] have developed the profound and elegant theory. We explain here one of their results. We develop $s^m P(s^{-1}\xi+\zeta)$ in ascending power of s:

$$s^m P(s^{-1}\xi+\zeta) = s^p\ P_\xi(\zeta) + O(s^{p+1}) \tag{2.2}$$

where $P_\xi(\zeta) \neq 0$. The number $p = m_\xi(P)$ is the multiplicity of ξ relative to P and the polynomial $P_\xi(\zeta)$ is called the localization of P at ξ. Then one can prove that $P_\xi(\zeta)$ is also hyperbolic with respect to ϑ. (2.2) means that $P_\xi(\zeta)$ is the best approximation of $P(\zeta)$ at a point ξ. If A has no singular point, $P_\xi(\zeta)$ is a polynomial of at most first degree as stated in the above. Atiyah-Bott-Gårding have proved the following

Theorem 2. $\lim_{s\to\infty} s^{m-p}\ e^{-is\langle x,\xi\rangle}\ E(P)(x) = E(P_\xi)(x)$

in the space of distributions, and

$$\bigcup_{\xi\neq 0} S\ E(P_\xi) \subset SS\ E(P) \subset \bigcup_{\xi \neq 0} \mathrm{c.h.}S\ E(P_\xi)$$

where c.h.A is the convex hull of a set A.

One of their conjectures is

$$SS\ E(P) = \bigcup_{\xi\neq 0} S\ E(P_\xi) \quad .$$

But this is not true (M. Tsuji [14]). We must amend the above formula a little in the following way. We develop $s^{m-p}\ P(s\xi+\zeta)^{-1}$ in the formal power series of $1/s$,

$$\frac{s^{m-p}}{P(s\xi+\zeta)} = \frac{1}{P_\xi(\zeta)} + \Sigma_{j=1}^{\infty} \frac{Q_j(\zeta)}{P_\xi(\zeta)^{j+1}} \left(\frac{1}{s}\right)^j \quad .$$

Here we put $E_0(P_\xi) = E(P_\xi)$ and

$$E_j(P_\xi) = (2\pi)^{-n} \int_{R^n - i\gamma\vartheta} e^{i<x,\zeta>}\ Q_j(\zeta)\ P_\xi(\zeta)^{-j-1}\ d\zeta \quad .$$

Theorem 3. Suppose that the multiplicity of $P(\xi)$ is at most double, then we get

$$SS\ E(P) = \bigcup_{\xi\neq 0} \bigcup_{j=0}^{\infty} S\ E_j(P_\xi) \quad . \tag{2.3}$$

Though the right hand side of (2.3) is abstract, we can exactly calculate it (M. Tsuji [14], [15]). As the calculation is long, we omit here the proof. In this case, though the description is concrete, it is a little complicated. Therefore we would like to give clear algebraic meaning to the formula (2.3). Taking care of the Remark of Theorem 1, we would like to use the notion of dual algebraic surface to express (2.3). Then we need to define

We quote here the definition of S. K. Kleiman [8].

Let Z be an algebraic variety and $x \in Z$ be a singular point of Z. A hypersurface is considered to be tangent at x if it is a limit of hypersurfaces tangent at smooth points approaching x. Then the dual variety of Z is defined as the closed envelope of tangent hypersurfaces.

If we accept this definition, the formula (2.3) is different of the dual algebraic surface of A. Therefore we would like t see whether or not it is possible to define a dual algebraic surface so that (2.3) coincides with the dual algebraic surface of A, or we would like to give some algebraic meaning to (2.3).

For the detailed surveys on the subject of this section, refer to G. F. D. Duff [4], L. Gårding [6] and L. Hörmander [7].

§3. Phase functions.

We put $\vartheta=(1,0,\dots,0)$ and assume that $P(D_x)$ is strictly hyperbolic with respect to ϑ, i.e.,

$$P(\xi) = \prod_{i=1}^{m}(\xi_1 - \lambda_i(\xi')) \quad , \ \xi'=(\xi_2,\dots,\xi_n) \in R^{n-1} \ ,$$

where $\lambda_i(\xi') \neq \lambda_j(\xi')$ if $i\neq j$ and $\xi'\neq 0$. In this case, $E(P)(x)$ is written down as

$$E(P)(x) = \sum_{j=1}^{m} \int_{R^{n-1}} a_j(\xi')\, e^{i\phi_j(x,\xi')}\, d\xi' \quad ,$$

where $\phi_j(x,\xi') = x_1\lambda_j(\xi') + \langle x',\xi'\rangle$ and $\langle x',\xi'\rangle = \Sigma_{j=2}^{n} x_j\xi_j$. The function $\phi_j(x,\xi')$ is called "phase function" and it satisfies the partial differential equation of first order as follows:

(3.1) $$\frac{\partial \phi_j}{\partial x_1} + \lambda_j\left(\frac{\partial \phi_j}{\partial x'}\right) = 0 \quad ,$$

(3.2) $$\phi_j(0,x') = \langle x',\xi'\rangle \quad .$$

The equation (3.1) is called "eikonal equation". For strictly hyperbolic operator $P(x,D_x)$ with variable coefficients, the elementary solution can be locally written down in the similar form. Let

$$p(x,\xi) = \Pi_{i=1}^{m} (\xi_1 - \lambda_i(x,\xi')) \quad ,$$

then it holds

(3.3) $$E(P)(x) = \sum_{j=1}^{m} \int_{R^{n-1}} a_j(x,\xi')\, e^{i\phi_j(x,\xi')}\, d\xi' \qquad \text{modulo} \quad C^\infty\text{-functions}$$

where $\phi_j(x,\xi')$ is the solution of

(3.4) $$\frac{\partial \phi_j}{\partial x_1} + \lambda_j\left(x,\frac{\partial \phi_j}{\partial x_2}, \ldots ,\frac{\partial \phi_j}{\partial x_n}\right) = 0 \quad ,$$

(3.5) $$\phi_j(0,x') = \langle x',\xi'\rangle \quad .$$

The reason why the representation (3.3) is local is that the Cauchy problem (3.4)-(3.5) can not admit a global smooth solution as a function of $x=(x_1, \ldots ,x_n)$. But (3.3) shows that the phase function $\phi_j(x,\xi')$ can be regarded as a function defined on the cotangent space. Moreover, as the equation (3.4) is of Hamilton-Jacobi type, the solution surface is composed of the family of hamiltonian flows. Therefore we can discuss the global construction of phase functions in the framework of symplectic geometry. This is the theory of Fourier integral operators and also Maslov's theory. For this subject, refer to L. Hörmander [7] and V. P. Maslov [11]. In [7], one can find the detailed bibliography.

In the mathematical physics, we meet very often the equation of Hamilton-Jacobi type. In the applied problems, the solutions must generally be single valued functions defined on the base space, not on the cotangent space. To get the solutions of this kind, the above discussions suggest us to take the projection of Lagrangean manifold onto the base space. This is the subject treated in the next section.

§4. General partial differential equations of first order.

We consider the Cauchy problem for partial differential equation of first order as follows:

$$(4.1) \qquad \frac{\partial u}{\partial t} + f(t,x,u,\frac{\partial u}{\partial x}) = 0 \qquad \text{in} \quad \{t>0,\ x\in R^n\} \quad ,$$

$$(4.2) \qquad u(0,x) = \phi(x) \qquad \text{on} \quad \{t=0,\ x\in R^n\}$$

where $f\in C^\infty(R^1\times R^n\times R^1\times R^n)$ and $\phi\in C^\infty(R^n)$. It is well known that the Cauchy problem (4.1)-(4.2) has uniquely a C^∞-solution in a neighborhood of t=0, and also that the problem (4.1)-(4.2) can not have, in general, global C^1-solution. This means that, when one prolongs the smooth solutions for large time, the singularities may appear. Therefore we consider the generalized solutions which contains the singularities. Our interest is to see the structure of singularities of generalized solutions.

On the other hand, the global existence of generalized solutions has been well studied from the various points of view. For the detailed bibliography, refer to P. D. Lax [9], P.L. Lions [10] and J. Smoller [13]. The most well-known method to prove the existence of solutions is the viscosity method. At first one consider the Cauchy problem for nonlinear parabolic equation:

$$(4.3)\quad \begin{cases} \dfrac{\partial}{\partial t}u_\varepsilon + f(t,x,u_\varepsilon,\dfrac{\partial}{\partial x}u_\varepsilon) = \varepsilon\ \Delta u_\varepsilon \qquad (\varepsilon>0) \quad , \\ \\ u_\varepsilon(0,x) = \quad \phi(x) \quad . \end{cases}$$

Using the estimates for linear parabolic equation, one can get the global solution $u_\varepsilon(t,x)$ of (4.3) which is smooth on the whole space. Next we make ε go to 0, then $\{u_\varepsilon(t,x)\}$ converges to $u(t,x)$ which is a generalized solution of (4.1)-(4.2). We must pay attention to the fact that, though $u_\varepsilon(t,x)$ is smooth, the limit $u(t,x)$ may contain the singularities. This means that it is a little difficult to get the informations on the singularities of $u(t,x)$ by the analysis of $\{u_\varepsilon(t,x)\}$. Therefore we try to solve the Cauchy problem (4.1)-(4.2) exactly by the characteristic method. Then the characteristic equations are written as follows:

$$
(4.4)\quad \begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial f}{\partial p_i}(t,x,v,p) \qquad (i=1,2,\ldots,n) \\[2ex] \dfrac{dv}{dt} = \Sigma_{j=1}^{n} \dfrac{\partial f}{\partial p_j}(t,x,v,p)\ p_j - f(t,x,v,p) \\[2ex] \dfrac{dp_i}{dt} = -\dfrac{\partial f}{\partial x_i}(t,x,v,p) - \dfrac{\partial f}{\partial v}(t,x,v,p)\ p_i \qquad (i=1,2,\ldots,n) \end{cases}
$$

$$
(4.5)\quad x_i(0) = y_i\ ,\quad v(0) = \phi(y)\ ,\quad p_i(0) = \frac{\partial \phi}{\partial y_i}(y)\quad .
$$

We assume here the following hypothesis:

(H) The Cauchy problem (4.4)-(4.5) has a global solution $x=x(t,y)$, $v=v(t,y)$ and $p=p(t,y)$ with respect to t for all $y \in R^n$.

As $Dx/Dy(0,y) = 1$ for all $y \in R^n$ where $Dx/Dy(t,y)$ is the Jacobian of C^∞-mapping $x=x(t,y)$, there exists an open neighborhood Ω of $t=0$ where $Dx/Dy(t,y) \neq 0$. By the theorem of inverse functions, we can uniquely solve the equation $x=x(t,y)$ with respect to y in Ω and write it by $y=y(t,x)$. Put $u(t,x) = v(t,y(t,x))$, then $u(t,x)$ satisfies the equation (4.1)-(4.2) in a neighborhood of $t=0$. This is the classical theorem on the local existence of smooth solution for (4.1)-(4.2). But it is natural that the Jacobian vanishes at some points. If we solve the equation $x=x(t,y)$ with respect to y in a neighborhood of a point where the Jacobian vanishes, we see that the inverse function becomes many-valued. We write it by $y=g_i(t,x)$ $(i=1,2,..,k)$, then the solution of (4.1) takes several values $\{u_i(t,x);i=1,..,k\}$

where $u_i(t,x)=v(t,g_i(t,x))$. When the space dimension $n=2$, we benefit here very much the results of H. Whitney [18].

As we are generally asked to get single valued solutions in the applied mathematics, we must choose only one value from $\{u_i(t,x)\}$ so that it satisfies some additional conditions. These conditions depend on the type of equations. For example, see M. Tsuji [16] for Hamilton-Jacobi equation and S. Nakane [12] for the equation of conservation law. The generalized solutions for Hamilton-Jacobi equation are generally Lipschitz continuous, and the weak solutions for the equations of conservation law are piecewise smooth. For the reason why such a difference may happen, see Tsuji [17]. If we choose only one appropriate value from $\{u_i(t,x)\}$, we see that the solution contains necessarily the singularities.

Summing up the above discussion, we can say that our proof is principally divided into two parts. The first one is to solve the equation $x=x(t,y)$ with respect to y in a neighborhood of a singular point. This is the application of the theory of the singularities of smooth mappings. The second part is to choose only one value so that the solution becomes a single valued function. This is the problem of analysis.

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Lagrangian stability of a Lagrangian map-germ in a restricted class —— open Whitney umbra and open swallowtails

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0. This is a summary of recent results obtained by the author.

1. Let $g : \mathbb{R}^m,0 \longrightarrow \mathbb{R}^n,0$ be a C^∞ map-germ. Denote by $g^* : E_n \longrightarrow E_m$ the induced $\mathbb{R}$-algebra homomorphism, where $E_n = \{h : \mathbb{R}^n,0 \longrightarrow \mathbb{R}\ \ C^\infty \text{ function-germs}\}$.

Define g^*E_n-subalgebra H_g of E_m by

$$H_g = \{k \in E_m \mid dk \in \textstyle\sum_{i=1}^{n} dg_i \cdot E_m\},$$

where d means the exterior differential.

In the case $m > n$, the algebra H_g is studied by Moussu, Tougeron [6] and Malgrange [5], related to the characterization of composite differentiable functions and the structure of relative de Rham cohomologies of map-germs.

On the other hand, the algebra H_g naturally appears in the singularity theory in symplectic geometry, especially in the case

$m \leq n$. In this paper, we restrict ourselves to the most important case $m = n$ and we will characterize Lagrangian stability in a restricted class, utilizing H_g or its analogue.

We set $H_g^{(-1)} = E_n$, $H_g^{(0)} = H_g$ and inductively

$$H_g^{(i)} = \{k \in E_n \mid dk \in \sum_{i=1}^n dg_i \cdot H_g^{(i-1)}\},$$

$(i = 0,1,2,\ldots)$. Then we have a sequence of g^*E_n-subalgebras:

$$E_n = H_g^{(-1)} \supset H_g^{(0)} \supset H_g^{(1)} \supset \cdots \supset g^*E_n.$$

2. In classical machanics, the canonical one-form θ and the symplectic two-form $\omega = d\theta$ on the cotangent bundle over a configuration space play basic roles [1]. Notice that, for a canonical coordinate $p_1,\ldots,p_n$, $q_1,\ldots,q_n$ on the cotangent bundle $T^*\mathbb{R}^n$ of $\mathbb{R}^n$,

$$\theta = \sum_{i=1}^n p_i dq_i, \quad \omega = \sum_{i=1}^n dp_i \wedge dq_i.$$

Definition 2.1. A C^∞ map-germ $f : \mathbb{R}^n,0 \longrightarrow T^*\mathbb{R}^n$ is called <u>Lagrangian</u> if $f^*\omega = 0$. We denote by L_n the set of all Lagrangian map-germs $\mathbb{R}^n,0 \longrightarrow T^*\mathbb{R}^n$.

Let $f \in L_n$. Then the one-form $f^*\theta$ is closed and therefore exact. Hence, there is unique $e \in E_n$ up to constant addition such that $de = f^*\theta$, which is called a <u>generating function</u> of f.

Remark that H_g is equal to the set of generating functions of Lagrangian liftings of g.

Example 2.2. (1) Define $f : \mathbb{R}^2,0 \longrightarrow T^*\mathbb{R}^2$ by

$$f(x,y) = (q_1 \circ f,\ q_2 \circ f,\ p_1 \circ f,\ p_2 \circ f) = (x,\ y^3 + xy,\ -(1/2)y^2,\ y).$$

Then f is a Lagrangian immersion. $e = (3/4)y^4 + (1/2)xy^2$ is a generating function of f.

(2) Define $f : \mathbb{R}^2,0 \longrightarrow T^*\mathbb{R}^2$ by

$$f = (x,\ y^3 + xy,\ -(3/10)y^5 - (1/6)xy^3,\ (3/4)y^4 + (1/2)xy^2).$$

Then f is a Lagrangian map-germ and its image is called the <u>open swallowtail</u>, which is introduced by Arnol'd [3]. We call also f by the same name.

(3) Define $f : \mathbb{R}^2,0 \longrightarrow T^*\mathbb{R}^2$ by

$$f = (x,\ y^2,\ (2/3)y^3,\ xy).$$

Then f is a Lagrangian map-germ and f is called the <u>open Whitney umbrella</u> (Givental' [4]).

Let $\pi : T^*\mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the projection, that is, $\pi(p,q) = q$.

Definition 2.3.(Arnol'd [2]). (1) Two Lagrangian map-germs f and $f' : \mathbb{R}^n,0 \longrightarrow T^*\mathbb{R}^n$ are called <u>Lagrange equivalent</u> if there exist diffeomorphism-germs $\psi : \mathbb{R}^n,0 \longrightarrow \mathbb{R}^n,0$, $\Phi : T^*\mathbb{R}^n,f(0) \longrightarrow T^*\mathbb{R}^n,f'(0)$ and $\varphi : \mathbb{R}^n,\pi(f(0)) \longrightarrow \mathbb{R}^n,\pi(f'(0))$

such that $\Phi^*\omega = \omega$, $\pi\circ\Phi = \varphi\circ\pi$ and that $\Phi\circ f = f'\circ\psi$.

(2) Let $S \subset L_n$ be a subset. A Lagrangian map-germ $f \in S$ is called Lagrange stable in S if any unfolding $F : \mathbb{R}^n\times \mathbb{R}^r,0 \longrightarrow T^*\mathbb{R}^n\times \mathbb{R}^r$ of f by Lagrangian map-germs in S is trivialized with respect to Lagrangian equivalence. An $f \in L_n$ is called Lagrange stable if f is Lagrange stable in L_n.

We fix a C^∞ map-germ $g : \mathbb{R}^n,0 \longrightarrow \mathbb{R}^n,0$ and consider the following classes of Lagrangian map-germs related to $H_g^{(i)}$. We set

$$L_g^{(i)} = \{f \in L_n \mid \pi\circ f = g,\ p_j\circ f \in H_g^{(i-1)},\ (j = 1,\dots,n)\},$$

$(i = 0,1,2,\dots)$.

Remark 2.4. (1) If $f \in L_g^{(i)}$ and e is a generating function of f, then $e \in H_g^{(i)}$.

(2) If g is stable in the sense of Thom-Mather and $f \in L_g^{(0)}$ is Lagrange stable in $L_g^{(0)}$, then f is Lagrange stable.

To state our result, we recall the following notion.

A pair (ξ,η) of germs of vector-fields over $(\mathbb{R}^n,0)$ is called g-compatible if $Tg\circ\xi = \eta\circ g$. In this case ξ (resp. η) is called g-lowerable (resp. g-liftable).

Now we have

Theorem A. Let $g : \mathbb{R}^n,0 \longrightarrow \mathbb{R}^n,0$ be a finite analytic map-germ, $f \in L_g^{(i)}$ and $e \in H_g^{(i)}$ be a generating function of f, $(i = 0,1,2,\ldots)$. Then the following conditions are equivalent:

(I) f is Lagrange stable in $L_g^{(i)}$.

(II) $H_g^{(i)} = \{\xi e + h \circ g \mid \xi$ is g-lowerable, $h \in E_n\}$.

(III) $H_g^{(i)} = \{ \sum_{j=1}^n (\eta_j \circ g)(p_j \circ f) + \eta_0 \circ g \mid$

$\eta_j \in E_n$, $(0 \le j \le n)$, $\sum_{j=1}^n \eta_j (\partial/\partial q_j)$ is g-liftable$\}$.

Remark 2.5. A g-lowerable vector field $\xi : E_n \longrightarrow E_n$ maps $H_g^{(i)}$ into $H_g^{(i)}$, $(i = 0,1,2,\ldots)$.

Definition 2.6. An elememt e of $H_g^{(i)}$ is called <u>stable</u> <u>in</u> $H_g^{(i)}$ if the condition (II) of Theorem A is satisfied.

3. The case of kernel rank one.

Let $g : \mathbb{R}^n,0 \longrightarrow \mathbb{R}^n,0$ be of form $g = (x', \varphi(x',x_n))$, where $x' = (x_1,\ldots,x_{n-1})$.

Lemma 3.1. Let $\psi \in H_g^{(i)}$, $(i = -1,0,1,\ldots)$. Then there exists $f_\psi \in L_g^{(i+1)}$, which is unique up to Lagrangian equivalence, such that $p_n \circ f_\psi = \psi$. Precisely, we can take

$$p_j \circ f = \int_0^{x_n} ((\partial\psi/\partial x_j)(\partial\varphi/\partial x_n)-(\partial\psi/\partial x_n)(\partial\varphi/\partial x_j))dx_n .$$

In this case, $e = \int_0^{x_n} \psi(\partial\varphi/\partial x_n)dx_n$ is a generating function of f_ψ.

Theorem B. Assume $g = (x',\varphi(x',x_n)) : \mathbb{R}^n,0 \longrightarrow \mathbb{R}^n,0$ is analytic and finite. If ψ is stable in $H_g^{(i)}$, $(i = -1,0,1,\ldots)$, then f_ψ is Lagrange stable in $L_g^{(i+1)}$. Especially, if ψ is stable in $H_g^{(i)}$, then $\int_0^{x_n} \psi(\partial\varphi/\partial x_n)dx_n$ is stable in $H_g^{(i+1)}$.

Corollary 3.2. The open Whitney umbrella is Lagrange stable. The open swallowtail is Lagrange stable in $L_g^{(1)}$, where $g = (x, y^3 + xy)$.

Remark 3.3. (1) Let $g = (x, y^2)$. Then H_g is generated by 1 and y^3 as E_2-module via g^*.

(2) Let $g = (x, y^3 + xy)$. Then H_g is generated by 1, $(3/4)y^4 + (1/2)xy^2$ and $(3/5)y^5 + (1/3)xy^3$ as E_2-module via g^*.

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COMPLETE INTEGRABLE FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS AND LEGENDRIAN UNFOLDINGS

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§0. Introduction

In this note we will announce some results about singularities of solutions of a first order partial differential equation. The singular point of a solution means that the solution has many valued near the point. In the classical theory, a first order partial differential equation (or, briefly, an equation) is written in the form

$$(*)\ F_k(x_1,\dots,x_n,z,p_1,\dots,p_n) = 0\ (k = 1,\dots,2n+1-r,\ r \geq n).$$

A (classical) solution of the equation $(*)$ is a smooth function $z = f(x_1,\dots,x_n)$ and $p_i = (\partial f/\partial x_i)(x)$. We assume that F_k are $(2n+1)$-variable smooth functions and

$$\mathrm{rank}(\partial F_k/\partial x_i, \partial F_k/\partial z, \partial F_k/\partial p_j) = 2n+1-r.$$

Define $D = \{(x,z) \mid$ there exists $p \in \mathbb{R}^n$ such that $F_1(x,z,p) = \cdots = F_{2n+1-r}(x,z,p) = 0$ and $\operatorname{rank}(\partial F_k/\partial p_j)(x,z,p) < \min(n, 2n+1-r)\}$. We call D a <u>discriminant set</u> of the equation (*). We also define $\Sigma = \{(x,z,p) \mid F_1(x,z,p) = \cdots = F_k(x,z,p) = 0$ and $\operatorname{rank}(\partial F_k/\partial p_j)(x,z,p) < \min(n,2n+1-r)\}$. We say that Σ is a <u>singular solution</u> of the equation (*) if D is a "graph" of a solution. But D is generally a hypersurface in $\mathbb{R}^n \times \mathbb{R}$ with singularities. Hence, we need a generalization of the notion of solutions. By the method of Lie, we consider the following system of equations :

$$\begin{cases} F_k(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0 \ (k = 1, \ldots, 2n+1-r,\ r \geqq n). \\ dz - \sum_{i=1}^n p_i dx_i = 0. \end{cases}$$

Namely we treat the distribution which is given by the 1-form $dz - \sum_{i=1}^n p_i dx_i$ on $\bigcap_{k=1}^{2n+1-r} F_k^{-1}(0)$. The <u>(abstract) solution</u> is defined to be a maximal integrable submanifold of such a distribution. For details, see Definition 1.3. We also consider the fibre structure of the projection $\pi(x,z,p) = (x,z)$. Even if we add this structure to our theory, there is an existence theorem of complete solutions.

DEFINITION 0.1. We say that an (r-n)-parameter family of (classical) solutions $z = f(x_1, \ldots, x_n, t_1, \ldots, t_{r-n})$ of the equation (*) is a <u>(classical) complete solution</u> if

$$\operatorname{rank}(\partial f/\partial t_i, \partial^2 f/\partial t_i \partial x_j) = r-n.$$

The following theorem is one of the highest result in the classical theory.

THEOREM 0.2. (<u>Classical existence theorem</u>). <u>If the equation (*) is involutory near a point</u> (x_0, z_0, p_0) <u>and</u> $(x_0, z_0, p_0) \notin \Sigma$, <u>then there exists a (classical) complete solution of (*) near</u> (x_0, z_0, p_0).

We say that the equation (*) is <u>involutory</u> if $[F_j, F_k] = 0$ $(1 \leqq j, k \leqq 2n+1-r)$, where

$[F,G] = F \cdot \partial G/\partial z - G \cdot \partial F/\partial z + \sum_{i=1}^{n} (\partial F/\partial x_i \cdot \partial G/\partial p_i - \partial F/\partial p_i \cdot \partial G/\partial x_i) + \sum_{i=1}^{n} p_i \cdot (\partial F/\partial z \cdot \partial G/\partial p_i - \partial G/\partial z \cdot \partial F/\partial p_i)$.

This result leads us to ask : What happens at a point of Σ?

In [3], we have studied the case where $r = 2n$ (i.e. the single equation case). We have shown that almost all single equation $F = 0$ has no singular solution and Σ consists of singular points of solutions. As the Clairaut type equation, there exist examples that there is a (classical) complete solution near Σ and Σ is a singular solution. Our theorem in [3] asserts that the Clairaut type equation is not generic in the space of all single equations. But the Clairaut type equation has been well-known from old time.

On the other hand, if we try to study the case where $r < 2n$, the involutory condition is very important as in Theorem 0.2. And involutory equations are not generic in the space of all equations.

According to these facts, we will restrict our attention to the category of equations with (abstract) complete solution. See Definition 1.7. Our results assert that the Clairaut type equation becomes to be generic in this category. The key result of our theory is Theorem 2.7 in §2. This theorem asserts that the space of equations with (abstract) complete solution is homeomorphic to an open subspace in the space of Legendrian immersions (i.e. the space of Legendrian unfoldings). By this theorem, we can translate generic properties of equations with (abstract) complete solution into corresponding generic properties of Legendrian unfoldings. By Arnol'd-Zakalyukin's theory ([1],[4]), we can study Legendrian unfoldings in terms of generating families.

In §3 we will introduce natural equivalence relations among equations with (abstract) complete solution. These equivalence relations describe bifurcations of singularities of (abstract) complete solutions and discriminant sets of equations. We will give lists of classifications of equations by these equivalence relations.

All maps considered here are differentiable of class C^{∞}, unless stated otherwise.

Detailed version will appear elsewhere.

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§1. Geometry of first order differential equations

The aim of this section is to describe the geometric structure connected with first order differential equations.

Let $J^1(\mathbb{R}^n,\mathbb{R})$ be the 1-jet bundle of functions of n-variables. Since we only consider local situations, the jet bundle $J^1(\mathbb{R}^n,\mathbb{R})$ may be considered as $\mathbb{R}^{2n+1}$ with natural coordinates given by $(x_1,\ldots,x_n,z,p_1,\ldots,p_n)$. We have a natural projection

$$\pi : J^1(\mathbb{R}^n,\mathbb{R}) \longrightarrow \mathbb{R}^n\times\mathbb{R}$$

which is defined by $\pi(x,z,p) = (x,z)$.

Let θ be the canonical contact form on $J^1(\mathbb{R}^n,\mathbb{R})$ which is given by $\theta = dz - \sum_{i=1}^n p_i dx_i$. Throughout the remainder of this paper, we shall consider $J^1(\mathbb{R}^n,\mathbb{R})$ as a contact manifold whose contact structure is given by the contact form θ. Using this approach, a first order partial differential equation is most naturally interpreted as being a closed subset of $J^1(\mathbb{R}^n,\mathbb{R})$. Unless the contrary is specifically stated, we use the following definition.

DEFINITION 1.1. A <u>system of first order partial differential equations (or, briefly an equation)</u> is an r-dimensional submanifold $E \subset J^1(\mathbb{R}^n,\mathbb{R})$, where $n \leq r \leq 2n$.

In order to define the notion of solutions of equations, we introduce the following notion.

DEFINITION 1.2. A submanifold $i : L \subset J^1(\mathbb{R}^n,\mathbb{R})$ is said to be a <u>Legendrian submanifold</u> if $\dim L = n$ and $i^*\theta = 0$. The image of $\pi\circ i$ is said to be a <u>wave front set</u> of i. It is denoted by $W(i)$.

By the philosophy of Lie, we may define the notion of solutions as follows.

DEFINITION 1.3. Let $E \subset J^1(\mathbb{R}^n,\mathbb{R})$ be an equation. An <u>(abstract) solution</u> of E is a Legendrian submanifold $i : L \subset J^1(\mathbb{R}^n,\mathbb{R})$ such that $i(L) \subset E$.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function. Then $j^1f : \mathbb{R}^n \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})$ is a Legendrian embedding. Hence, in our terminology, the (classical) solution of E is a smooth function f such that $j^1f(\mathbb{R}^n) \subset E$. On the other hand, we can show that an (abstract) solution $i : L \subset J^1(\mathbb{R}^n,\mathbb{R})$ is given by (at least locally) an jet extension j^1f of a smooth function f if and only if $\pi\circ i$ is a non-singular map. We now define the notion of "geometric" singularities of solutions.

DEFINITION 1.4. Let $i : L \subset J^1(\mathbb{R}^n,\mathbb{R})$ be a Legendrian submanifold. We say that $q \in L$ is a <u>Legendrian singular point</u> if $\operatorname{rank} d(\pi\circ i)_q < n$.

We also define the notion of singularities of equations.

DEFINITION 1.5. Le $E^r \subset J^1(\mathbb{R}^n,\mathbb{R})$ be an equation. Then $q \in E$ is

(1) a <u>contact singular point</u> if $\theta(T_qE) = 0$

and

(2) a <u>π-singular point</u> if rank $d(\pi|E)_q < \min(r,n+1)$.

Let $\Sigma(\pi|E)$ be the set of π-singular points. We say that $D_E = \pi(\Sigma(\pi|E))$ is the <u>discriminant set</u> of the equation E. the notion of singular solution is as follows.

DEFINITION 1.6. If the π-singular set $\Sigma(\pi|E)$ is a Legendrian submanifold, then we call it an <u>(abstract) singular solution</u> of the equation E. In this case, the discriminant set D_E is the graph of the (abstract) singular solution.

In the classical theory (cf §0), the notion of complete solutions is very important. Let $z = f(x_1,\dots,x_n,t_1,\dots,t_{r-n})$ be the (classical) complete solution of E, then we have a jet extension

$$j^1_*f : \mathbb{R}^n\times\mathbb{R}^{r-n} \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})$$

which is defined by $j^1_*f(x,t) = j^1f_t(x)$, where $f_t(x) = f(x,t)$. Then j^1_*f is an immersion if and only if

$$\operatorname{rank}(\partial f/\partial t_k, \partial^2 f/\partial x_i\partial t_k) = r-n.$$

Since dim E = r, then the above immersion gives (at least locally) a parametrization of E and $j^1_*f(\mathbb{R}^n\times t)$ is a (classical) solution of E for ant $t \in \mathbb{R}^{r-n}$. Hence, there exists a foliation on E whose leaves are classical solutions. Then the following definition is reasonable.

DEFINITION 1.7. Let $E \subset J^1(\mathbb{R}^n,\mathbb{R})$ be an equation. We say that E is complete integrable (or E has an (abstract) complete solution) if there exists an n-dimensional complete integrable distribution D on E such that $\theta_q(D_q) = 0$ for any $q \in E$.

By the Frobenius' theorem, we have the following proposition.

PROPOSITION 1.8. *Let* $E^r \subset J^1(\mathbb{R}^n,\mathbb{R})$ *be an equation. Then the following conditions are equivalent.*

(1) E *is complete integrable.*

(2) *For any* $q \in E$, *there exist a neighbourhood* U *of* q *in* E *and smooth functions* $\mu_1,\dots,\mu_{r-n}$ *on* U *such that* $d\mu_1\wedge \cdots \wedge d\mu_{r-n} \neq 0$ *on* U and

$$\langle d\mu_1,\dots,d\mu_{r-n}\rangle_{C^\infty(U)} \supset \langle\theta|E\rangle_{C^\infty(U)}$$

as $C^\infty(U)$-modules, where $C^\infty(U)$ denotes the ring of smooth functions on U.

(3) *For any* $q \in E$, *there exist a neighbourhood* $V\times W$ *of* 0 *in* $\mathbb{R}^n\times\mathbb{R}^{r-n}$ *and an embedding* $f : V\times W \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})$ *such that* $f(0) = q$, $f(V\times W) \subset E$ *and* $f|V\times t : V\times t \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})$ *are Legendrian embedding for any* $t \in W$.

§2. Complete integrable first order partial differential equations

In this section we will determine the framework of the class of equations which will be studied in the remainder of this paper. Since we will only study local properties, an equation is defined to be an immersion $f : U \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})$ where U is an open subset of $\mathbb{R}^n$. By Proposition 1.8, we may adopt the following definition.

DEFINITION 2.1. Let $f : U \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})$ be an equation. We say that f is <u>complete integrable</u> if there exists a submersion $\mu = (\mu_1,\ldots,\mu_{r-n}) : U \longrightarrow \mathbb{R}^{r-n}$ such that

$$\langle d\mu_1,\ldots,d\mu_{r-n}\rangle_{C^\infty(U)} \supset \langle f^*\theta\rangle_{C^\infty(U)}.$$

We call $\mu = (\mu_1,\ldots,\mu_{r-n})$ a <u>complete integral</u> of f and the pair $(f,\mu) : U \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n}$ is called a <u>first order partial differential equation with complete integral (or briefly, an equation with complete integral).</u>

REMARK. We can also define the above notions in terms of map germs : An <u>equation germ</u> is defined to be an immersion germ $f : (\mathbb{R}^r,0) \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})$. We say that f is <u>complete integrable</u> if there exists a submersion germ

$$\mu = (\mu_1,\ldots,\mu_{r-n}) : (\mathbb{R}^r,0) \longrightarrow \mathbb{R}^{r-n}$$

such that

$$\langle d\mu_1,\ldots,d\mu_{r-n}\rangle_{\mathcal{E}_r} \supset \langle f^*\theta\rangle_{\mathcal{E}_r}.$$

Then μ is called a <u>complete integral</u> of f and the pair $(f,\mu) : (\mathbb{R}^r,0) \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})$ is called an <u>equation germ with complete integral</u>.

By the above definition, we have the following simple lemma.

LEMMA 2.2. <u>Let</u> $(f,\mu) : U \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n}$ <u>be an equation with complete integral. Then there exist unique elements</u> $h_1,\dots,h_{r-n} \in C^\infty(U)$ <u>such that</u>

$$f^*\theta = \textstyle\sum_{i=1}^{r-n} h_i d\mu_i$$

<u>on</u> U.

We now consider the 1-jet space $J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R})$and the canonical 1-form Θ on the space. Let $(x_1,\dots,x_n,t_1,\dots,t_{r-n})$ be canonical coordinate systems on $\mathbb{R}^n\times\mathbb{R}^{r-n}$ and $(x_1,\dots,x_n,t_1\dots,t_{r-n},z,p_1,\dots,p_n,q_1,\dots,q_{r-n})$ be the corresponding coordinate systems on $J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R})$. Then the canonical 1-form is given by

$$\Theta = dz - \textstyle\sum_{i=1}^n p_i dx_i - \sum_{i=1}^{n-r} q_i dt_i = \theta - \alpha,$$

where $\alpha = \sum_{i=1}^{n-r} q_i dt_i$.

We define three natural projections. Namely,

$$\Pi_1 : J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R}) \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n},\ \Pi_1(x,t,z,p,q) = (x,z,p,t),$$

$$\Pi_2 : J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n} \longrightarrow \mathbb{R}^n\times\mathbb{R},\ \Pi_2(x,z,p,t) = (x,z),$$

$$\pi' : J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R}) \longrightarrow \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^{r-n},\ \pi'(x,t,z,p,q) = (x,z,t).$$

Then we have the following proposition.

PROPOSITION 2.3. (1) *Let* $(f,\mu) : U \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n}$ *be an equation with complete integral. Then there exists an unique Legendrian immersion* $\mathfrak{L} : U \longrightarrow J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R})$ *such that* $\Pi_1\circ\mathfrak{L} = (f,\mu)$.

(2) *Let* $\mathfrak{L} : U \longrightarrow J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R})$ *be a Legendrian immersion such that if* $\Pi_1\circ\mathfrak{L} = (f,\mu)$ *then* f *is an immersion and* μ *is a submersion. Then* (f,μ) *is an equation with complete integral.*

We denote $\mathfrak{L}_{(f,\mu)}$ the Legendrian immersion which is constructed in the above proposition. We have two corollaries of the above proposition.

COROLLARY 2.4. *Let* $(f,\mu) : U \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n}$ *be an equation with complete integral. Then* $\mathfrak{L}_{(f,\mu)}$ *is Legendrian non-singular if and only if* $\mu^{-1}(t)$ *is a (classical)solution for any* $t \in \mathbb{R}^{r-n}$.

COROLLARY 2.5. *Let* $(f,\mu) : U \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n}$ *be an equation with complete integral. For any* $u \in U$, *we denote* $\mathfrak{L}_{(f,\mu)} = (x_1\cdot f(u),\dots,x_n\circ f(u),\mu_1(u),\dots,\mu_{r-n}(u),$ $z\circ f(u),p_1\circ f(u),\dots,p_n\circ f(u),h_1(u),\dots,h_{r-n}(u))$ *by the local coordinate system of* $J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R})$. *Then* f *is contact singular at* $u_0 \in U$ *if and only if* $h_1(u_0) = \cdots = h_{r-n}(u_0)$.

Our purpose is to study the genericity for some properties of equations with complete integrals. Let $U \subset \mathbb{R}^r$ be an open set. We denote by $Int(U,J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n})$ the set of equations with complete integrals. We also define $L(U,J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R}))$ to be the set of Legendrian immersion germs $\mathfrak{L} : U \longrightarrow J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R})$ such that $\Pi_1\circ\mathfrak{L} = (f,\mu)$, where $f : U \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})$ is an immersion and $\mu : U \longrightarrow \mathbb{R}^{r-n}$ is a submersion. These sets are topological spaces equipped with the Whitney C^∞-topology. A subset of $\mathrm{Int}(U,J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n})$ (resp. $L(U,J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R}))$) is generic if it is an open and dense subset in $\mathrm{Int}(U,J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n})$ (resp. $L(U,J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R}))$).

DEFINITION 2.6. Let P be a property of equation germs with complete integral $(f,\mu) : (\mathbb{R}^r,0) \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n}$ (resp. Legendrian immersion germs $\mathfrak{L} : (\mathbb{R}^r,0) \longrightarrow J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R})$). The property P is generic if, for some neighbourhood U of 0 in $\mathbb{R}^r$, the set P(U) is generic subset in $\mathrm{Int}(U,J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n})$ (resp. $L(U,J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R}))$).

By Proposition 2.3, we have a well-defined continuous mapping

$$\Pi_{1*} : L(U,J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R})) \longrightarrow \mathrm{Int}(U,J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n})$$

defined by

$$\Pi_{1*}(\mathfrak{L}) = \Pi_1\circ\mathfrak{L}.$$

We can state the following theorem which is the key of our theory.

THEOREM 2.7. The continuous map

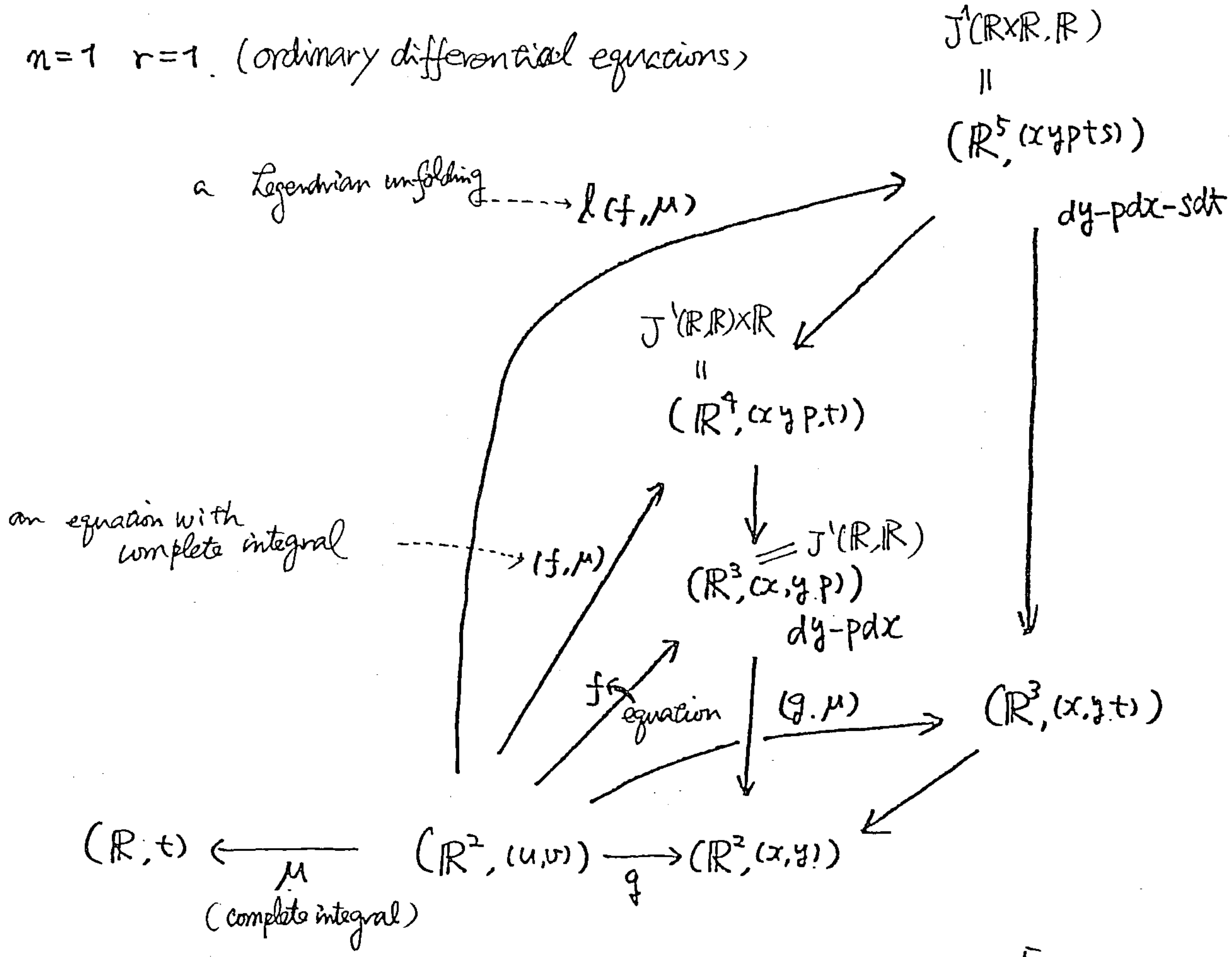
$$\Pi_{1*} : L(U,J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R})) \longrightarrow \mathrm{Int}(U,J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n})$$

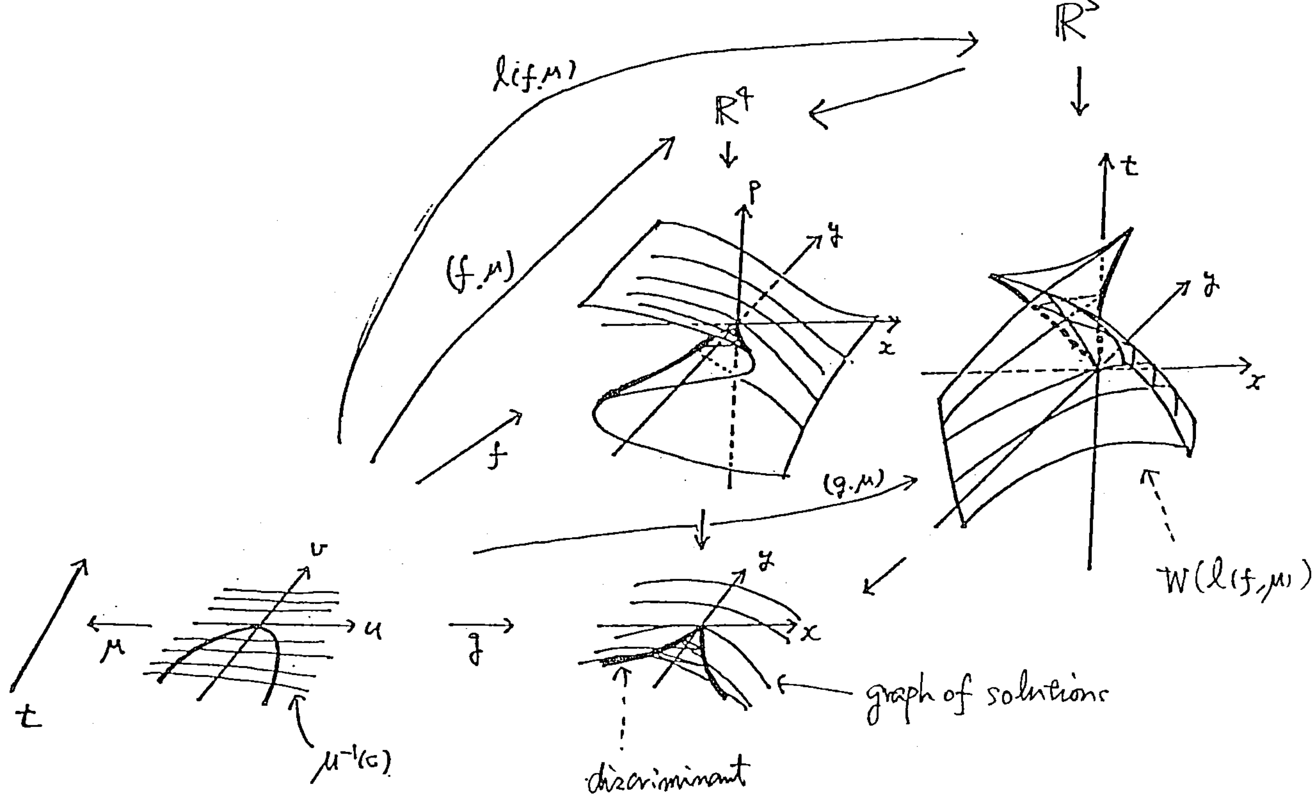
is a homeomorphism.

This theorem asserts that the genericity of a property of equations with complete integral can be interepreted by the genericity of the corresponding property of Legendrian immersions into $J^1(\mathbb{R}^n\times\mathbb{R}^{r-n},\mathbb{R})$. The Legendrian immersion $\mathcal{L}_{(f,\mu)}$ is called a <u>Legnedrian unfolding.</u>

On the other hand, by Arnol'd-Zakalyukin's theory ([1], [4]), we can study generic properties of Legendrian unfoldings in terms of generating families. These situation is described by the following diagram and figures :

$n=1 \quad r=1$. (ordinary differential equations)





§3. Some classifications

In this section we introduce three natural equivalence relations among equations with complete integrals.

Let $(f,\mu) : (\mathbb{R}^n,0) \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^{r-n}$ be an equation with complete integral. Our purpose is to study the discriminant set of $\pi\circ f$ and the bifurcation of Legendrian singularities of $\mu^{-1}(t)$ along the parameter $t \in (\mathbb{R}^{n-r},0)$. This situation leads us to the following definition.

DEFINITION 3.1. Let (g,μ) be a pair of a map-germ $g : (\mathbb{R}^r,0) \longrightarrow (\mathbb{R}^n\times\mathbb{R},0)$ and a submersion germ $\mu : (\mathbb{R}^r,0) \longrightarrow (\mathbb{R}^{r-n},0)$. Then the diagram

$$(\mathbb{R}^{r-n},0) \xleftarrow{\mu} (\mathbb{R}^r,0) \xrightarrow{g} (\mathbb{R}^n\times\mathbb{R},0),$$

(or briefly (g,μ)), is called an <u>integral diagram</u> if there exists an equation $f : (\mathbb{R}^r,0) \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})$ such that (f,μ) is an equation with complete integral and $\pi\circ f = g$.

Let $\pi_D : ((\mathbb{R}^n\times\mathbb{R})\times\mathbb{R}^{r-n},0) \longrightarrow (\mathbb{R}^n\times\mathbb{R},0)$ and $\pi_B : ((\mathbb{R}^n\times\mathbb{R})\times\mathbb{R}^{r-n},0) \longrightarrow (\mathbb{R}^{r-n},0)$ be canonical projections. We now introduce the following equivalence relations.

DEFINITION 3.2. Let (g,μ) and (g',μ') be integral diagrams. Then

(1) (g,μ) and (g',μ') are <u>bifurcation equivalent</u> (or, briefly <u>B-equivalent</u>) if the diagram

$$\begin{array}{ccccc}
(\mathbb{R}^r,0) & \xrightarrow{(g,\mu)} & ((\mathbb{R}^n\times\mathbb{R})\times\mathbb{R}^{r-n},0) & \xrightarrow{\pi_B} & (\mathbb{R}^{r-n},0) \\
\phi\downarrow & & \Psi\downarrow & & \psi\downarrow \\
(\mathbb{R}^r,0) & \xrightarrow[(g',\mu')]{} & ((\mathbb{R}^n\times\mathbb{R})\times\mathbb{R}^{r-n},0) & \xrightarrow[\pi_B]{} & (\mathbb{R}^{r-n},0)
\end{array}$$

commutes for some diffeomorphism germs ϕ, Ψ and ψ.

(2) (g,μ) and (g',μ') are <u>discriminant equivalent</u> (or, briefly <u>D-equivalent</u>) if the diagram

$$\begin{array}{ccccc}
(\mathbb{R}^r,0) & \xrightarrow{(g,\mu)} & ((\mathbb{R}^n\times\mathbb{R})\times\mathbb{R}^{r-n},0) & \xrightarrow{\pi_D} & (\mathbb{R}^n\times\mathbb{R},0) \\
\phi\downarrow & & \Psi\downarrow & & \psi\downarrow \\
(\mathbb{R}^r,0) & \xrightarrow[(g',\mu')]{} & ((\mathbb{R}^n\times\mathbb{R})\times\mathbb{R}^{r-n},0) & \xrightarrow[\pi_D]{} & (\mathbb{R}^n\times\mathbb{R},0)
\end{array}$$

commutes for some diffeomorphism germs ϕ, Ψ and ψ.

(3) (g,μ) and (g',μ') are <u>equivalent</u> if the diagram

$$\begin{array}{ccccc}
(\mathbb{R}^{r-n},0) & \xleftarrow{\mu} & (\mathbb{R}^r,0) & \xrightarrow{g} & (\mathbb{R}^n\times\mathbb{R},0) \\
\kappa\downarrow & & \psi\downarrow & & \phi\downarrow \\
(\mathbb{R}^{r-n},0) & \xleftarrow[\mu']{} & (\mathbb{R}^r,0) & \xrightarrow[g']{} & (\mathbb{R}^n\times\mathbb{R},0)
\end{array}$$

commutes for some diffeomorphism germs κ, ψ and ϕ.

What do these equivalence relations say? If (g,μ) and (g',μ') are B-equivalent, then $g|\mu^{-1}(t)$ and $g'|\mu^{-1}(\psi(t))$ are A-equivalent in the sense of Mather. Hence, the B-equivalence preserves bifurcations of Legendrian singularities of complete integrals along parameters. If (g,μ) and (g',μ') are D-equivalent, then g and g' are A-equivalent. Thus the D-equivalence preserves discriminant sets of equations. If (g,μ) and (g',μ') are equivalent, then these are B-equivalent and D-equivalent. Hence, the equivalence is the finest equivalence relation but it is hard to carry out the complete listing of the normal forms.

We will classify integral diagrams which are induced by generic equations with complete integrals by the above three equivalence relations. For our purpose, we can translate these equivalence into corresponding equivalence among Legendrian unfoldings. Especially B and D-equivalence are successfully classified in terms of generating families of Legendrian unfoldings. These caluculation will appear elsewhere. Our generic classifications are given in the following list

CANONICAL FORMS

I $n = 1$, $r = 2$. The ordinary differential equations case.

(Hayakawa, Ishikawa, Izumiya, Yamaguchi [2])

1) B-equivalence

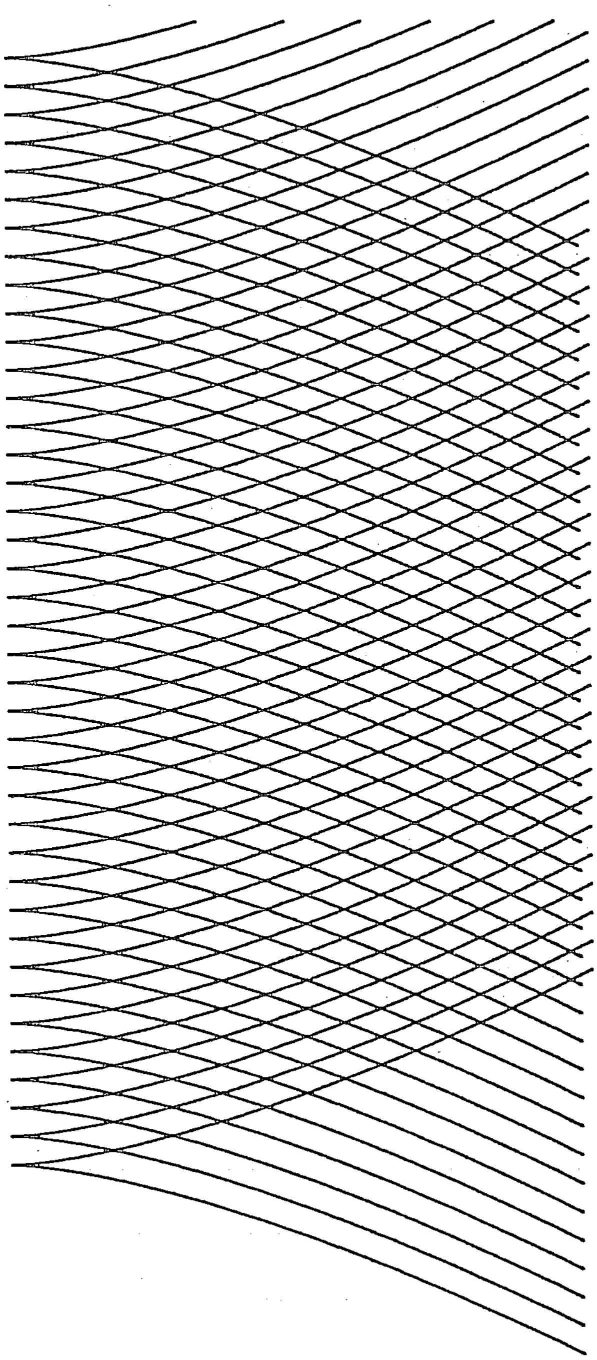
	$g\|\mu^{-1}(t)$
1)	$(u + t, u^2)$
2)	$(3u^2 + t, 2u^3)$
3)	$(4u^3 + t(2u + 1), 3u^4 + tu^2)$

2) D-equivalence

	g	μ	contact singular?
1)	(u,v)	v	regular
2)	(u^2,v)	$v - (1/3)u^3$	regular
3)	(u,v^2)	$v - (1/2)u$	singular
4)	$(u^3 + uv, v)$	$(3/4)u^4 + (1/2)u^2v + v$	regular
5)	$(u, v^3 + uv)$	v	singular
6)	$(u, v^3 + uv^2)$	$(1/2)v^2 + u$	singular

3) equivalence

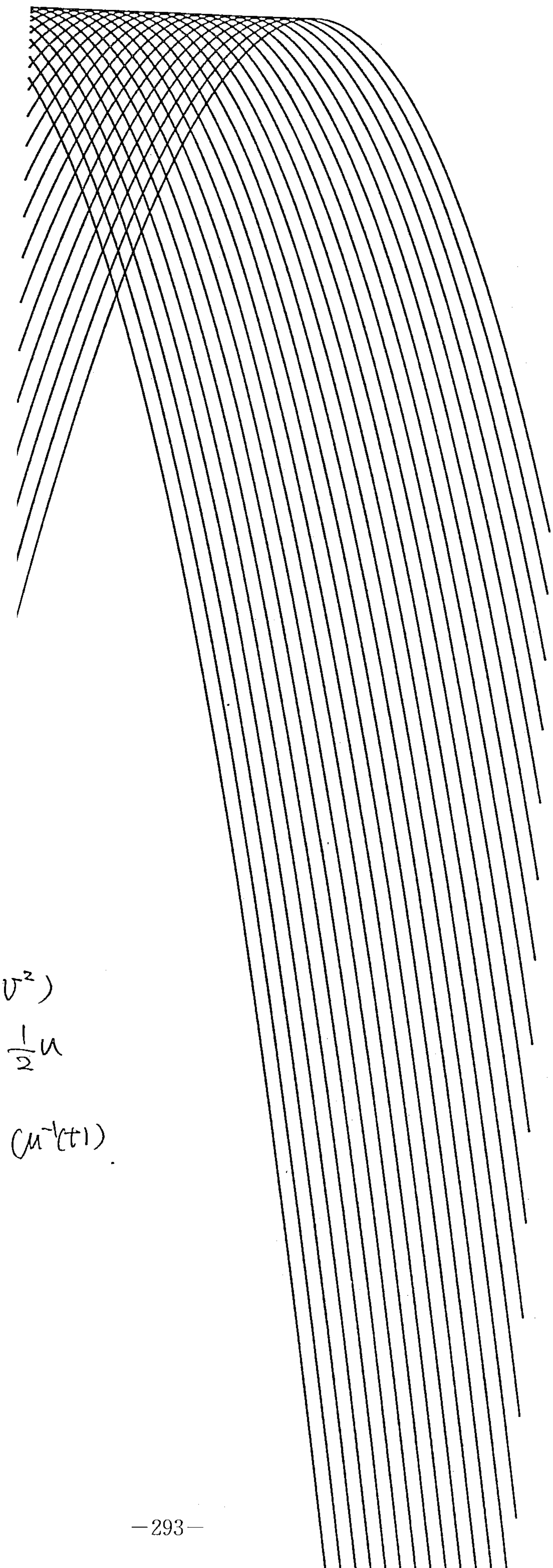
	g	μ
1)	(u,v)	v
2)	(u^2,v)	$v - (1/3)u^3$
3)	(u,v^2)	$v - (1/2)u$
4)	$(u^3 + uv, v)$	$(3/4)u^4 + (1/2)u^2v + \alpha\circ g$, $\alpha(0)=0$, $(\partial\alpha/\partial y)(0)=0$
5)	$(u, v^3 + uv)$	$v + \alpha\circ g$, $\alpha(0)=0$
6)	$(u, v^3 + uv^2)$	$(1/2)v^2 + \alpha\circ g$, $\alpha(0)=0$, $(\partial\alpha/\partial x)(0)=0$



$g(u,v) = (u^2, v)$

$\mu(u,v) = v - \frac{1}{3}u^3$

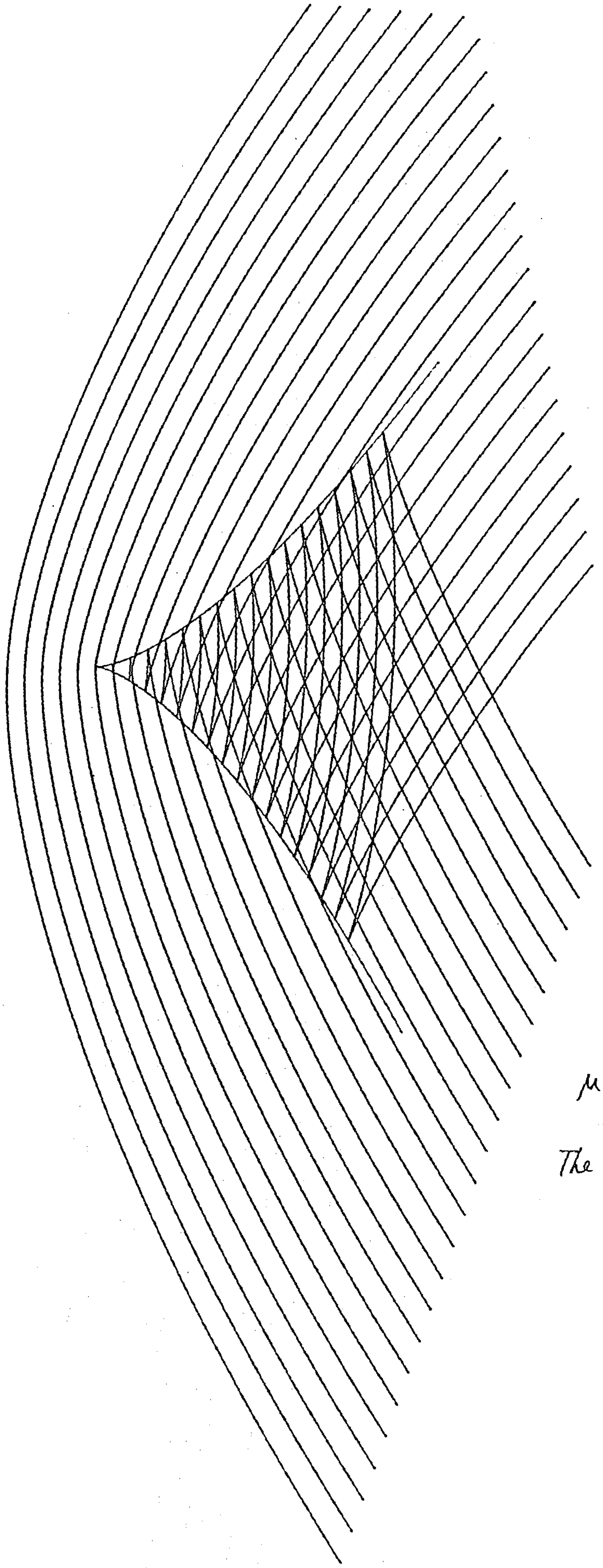
The figure of $g(\mu^{-1}(t))$.



$g(u,v) = (u, v^2)$

$\mu(u,v) = v - \frac{1}{2}u$

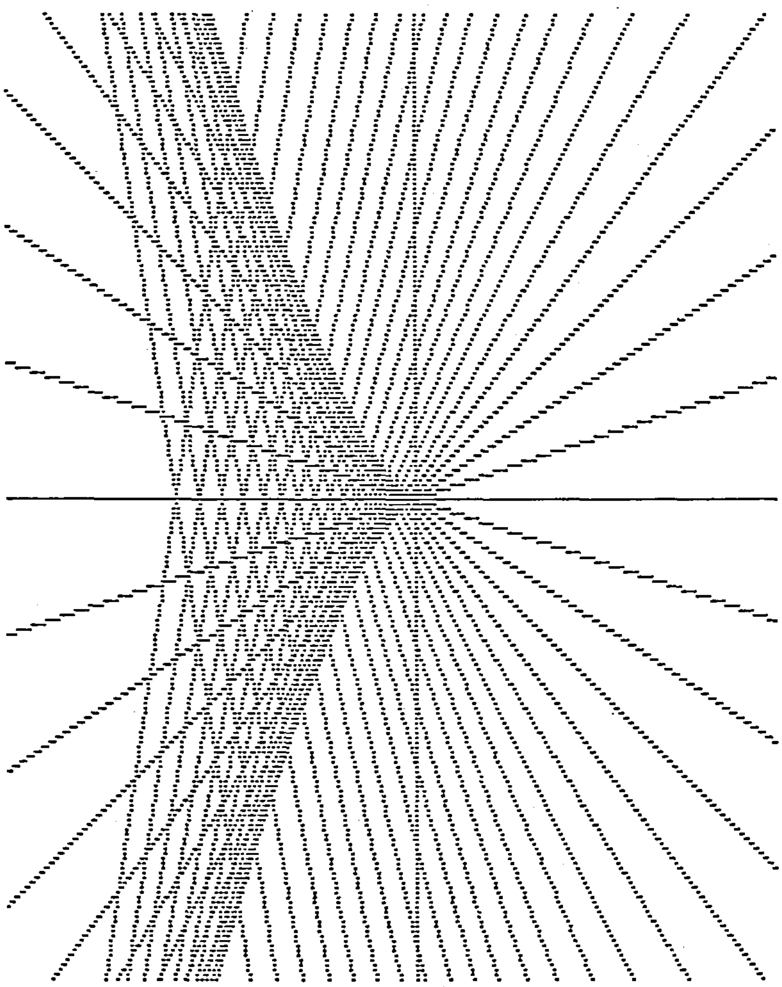
The figure of $g(\mu^{-1}(t))$.



$g(u,v) = (u^3+uv, v)$

$\mu(u\,v) = \frac{3}{4}u^4 + \frac{1}{2}u^2 v + v$

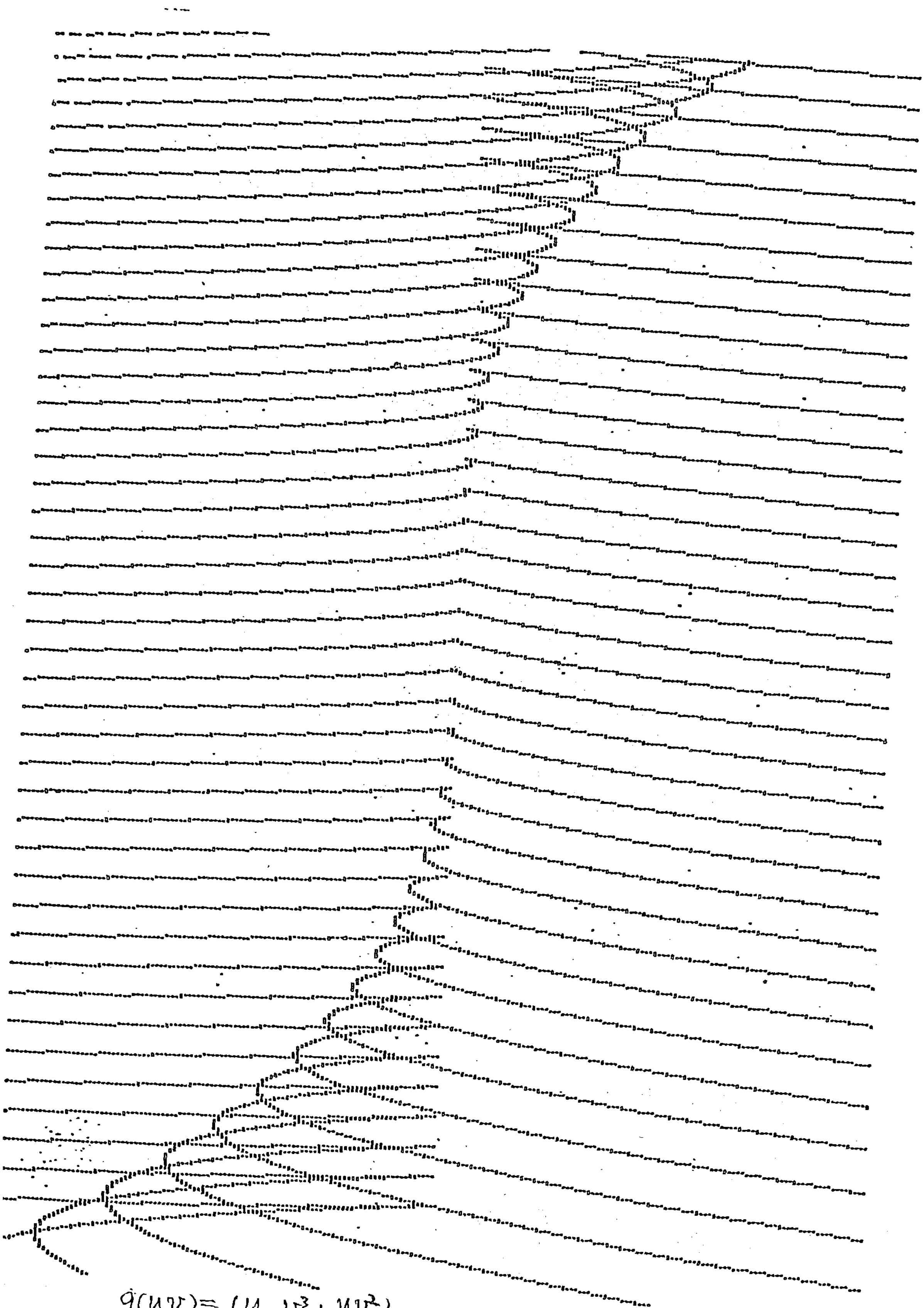
The figure of $g(\mu^{-1}(t))$.



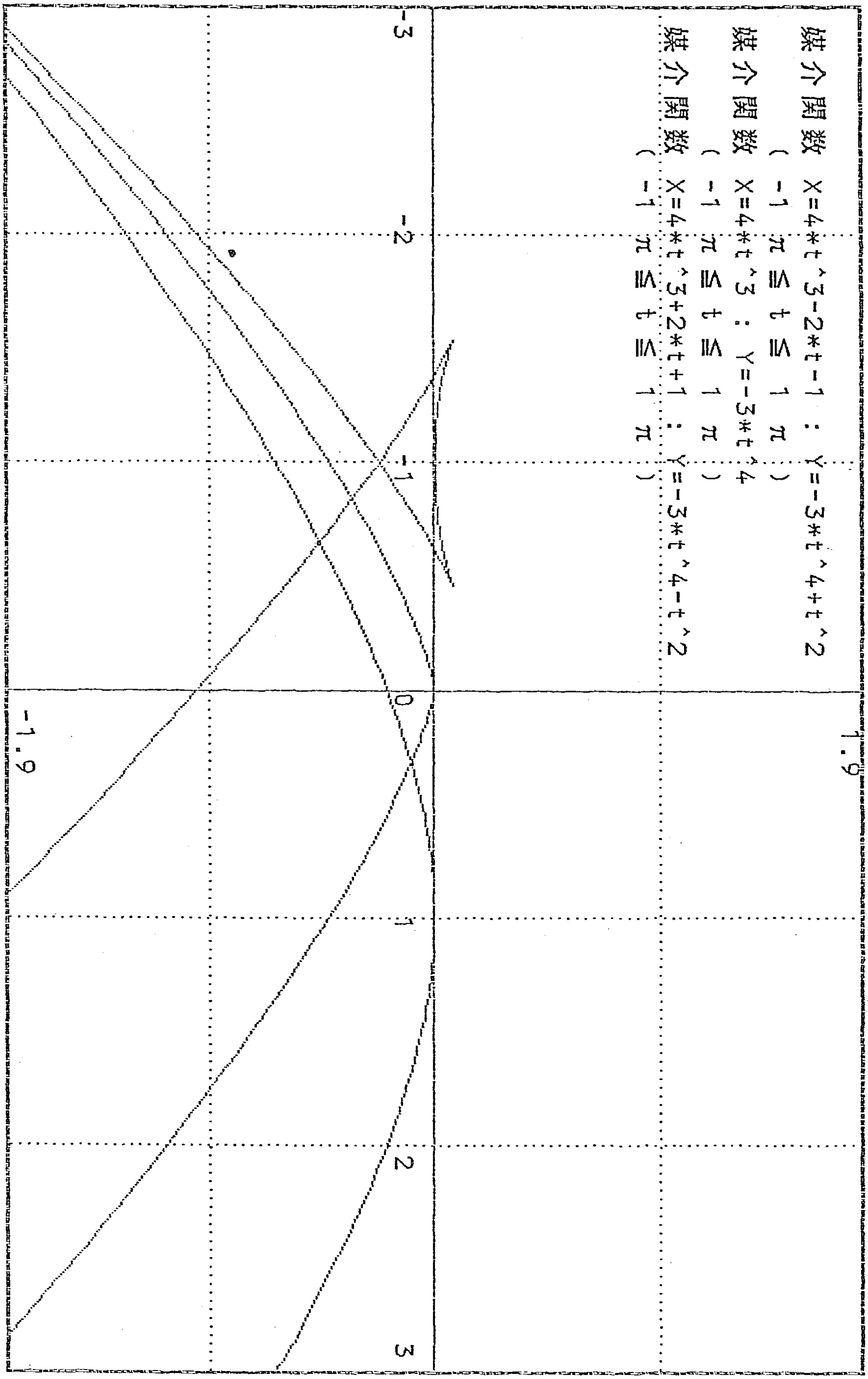
$g(uv) = (u, v^3 + uv)$

$\mu(uv) = v$

The figure of $g(\mu^{-1}(t))$.



$g(uv) = (u, v^3 + uv^2)$
$\mu(u.v) = \frac{1}{2}v^2 + u$
The figure of $g(\mu^{-1}(t))$



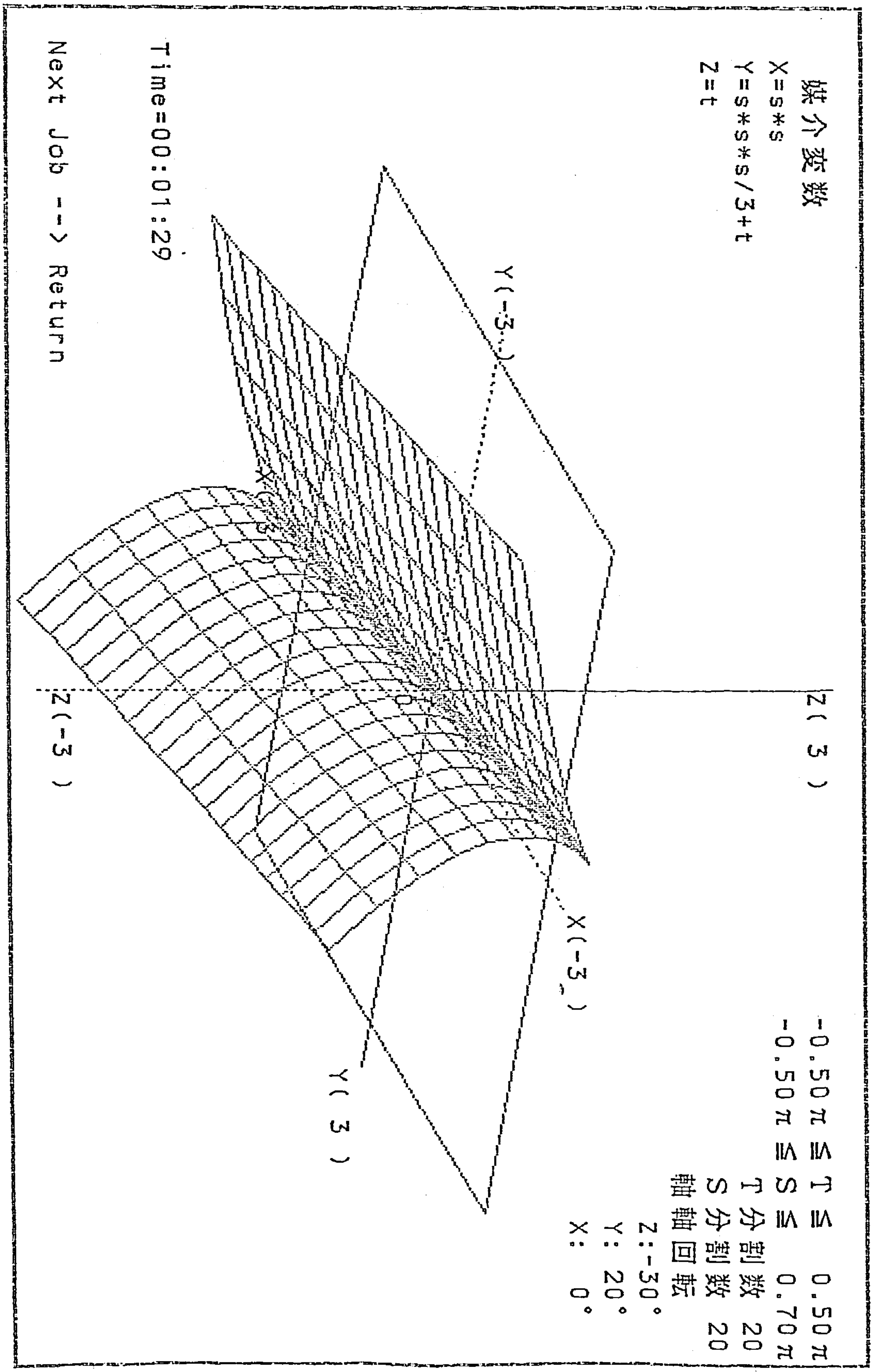
B-equivalence.

$g(\mu^{-1}(t)) = (4u^3 + t(2u+1),\ 3u^4 + tu^2)$

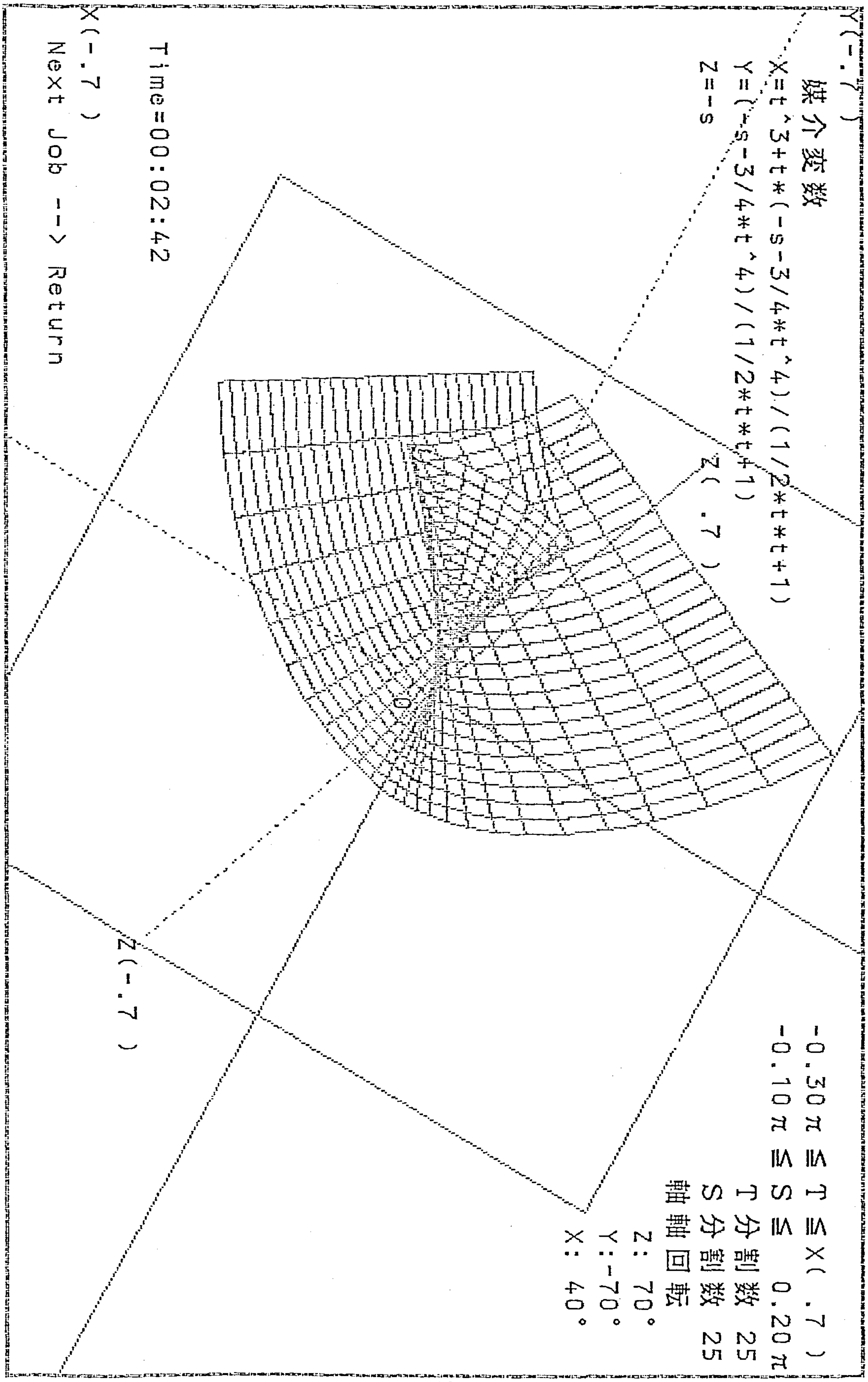
The figure of $W(l_{(f,\mu)})$ for

$g(u,v) = (u^2, v)$

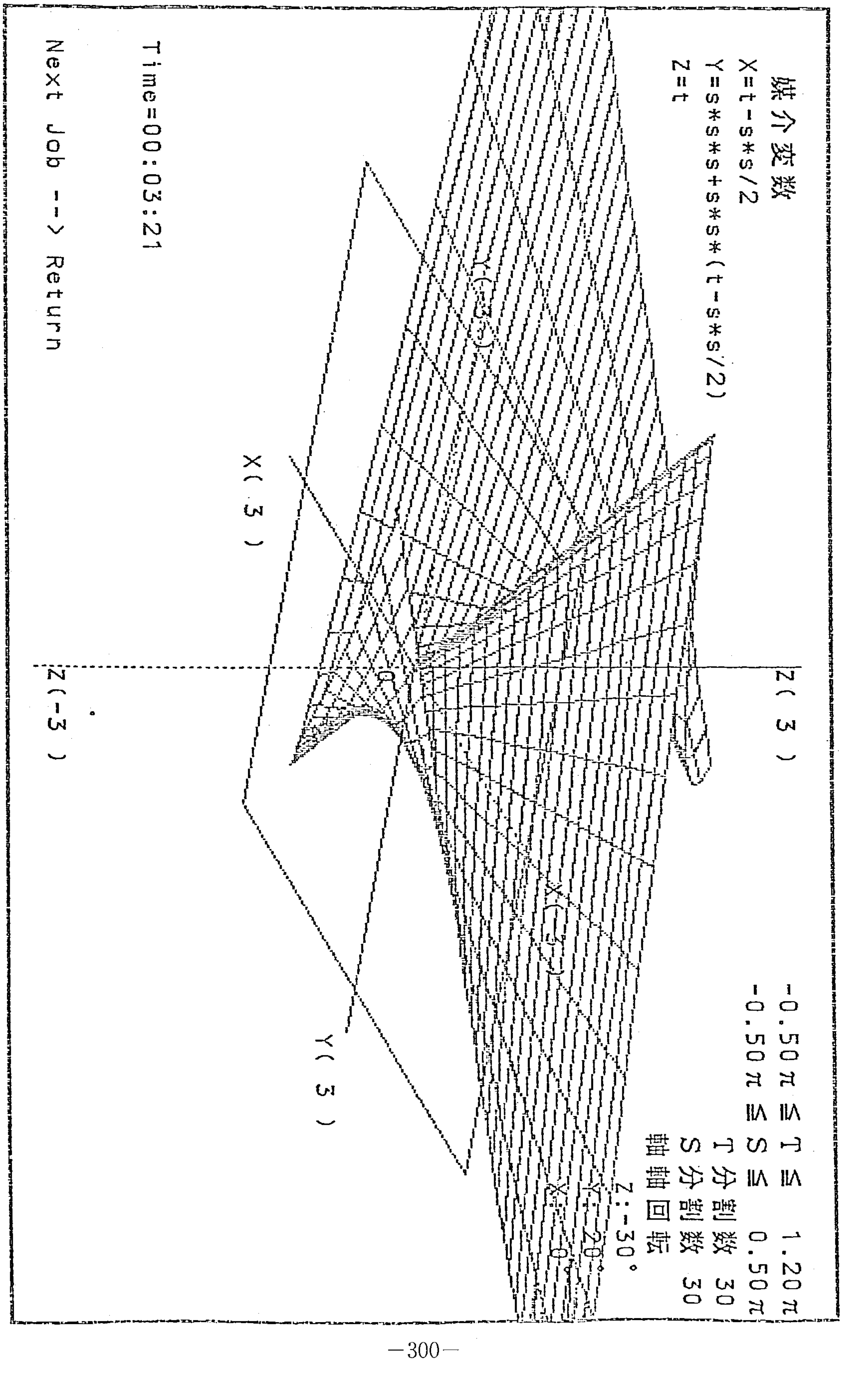
$\mu(u,v) = v - \frac{1}{3}u^3$.



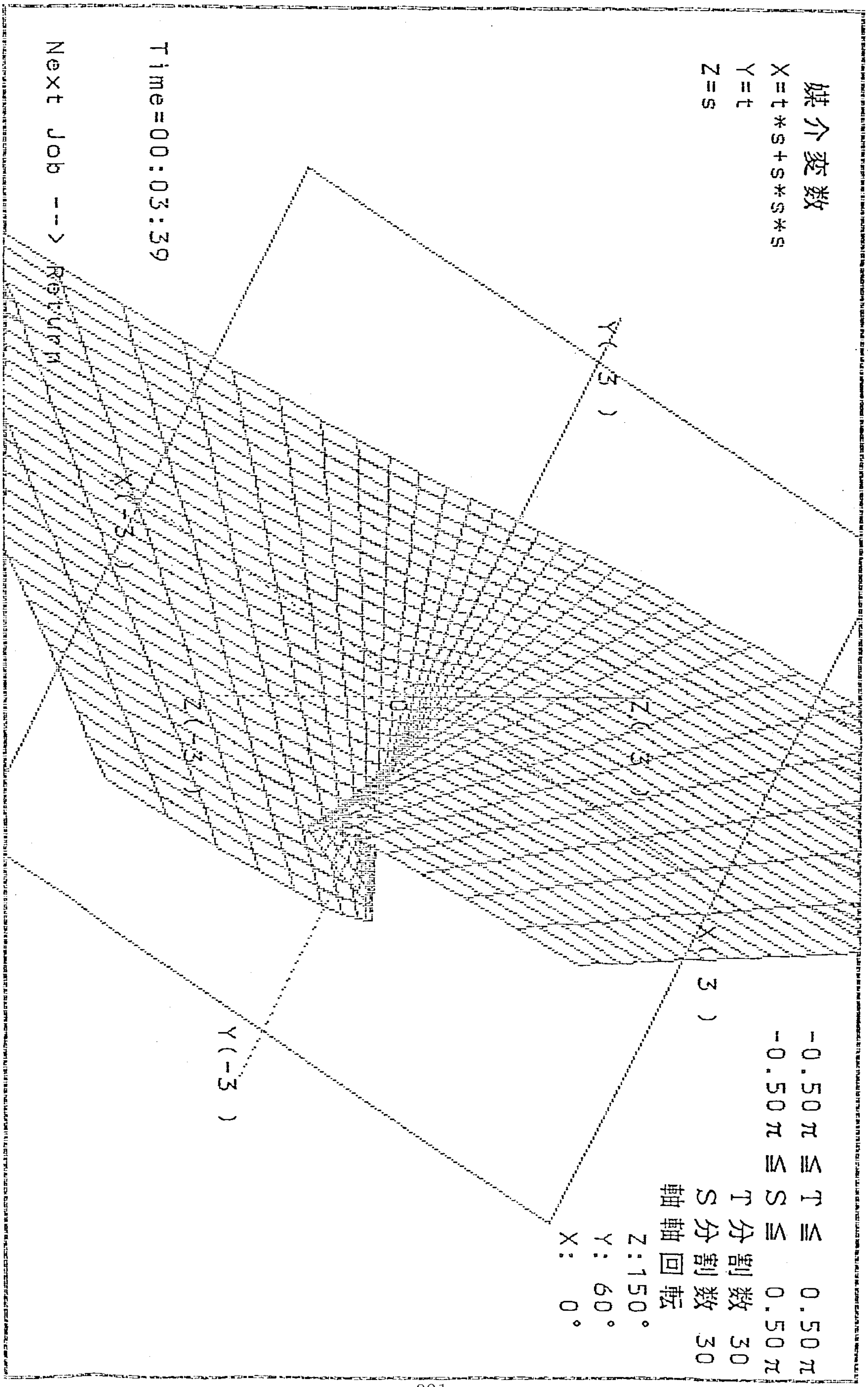
The figure of $W(\ell_{g,\mu})$ .. for
$g(u,v) = (u^3+uv, v)$
$\mu(u,v) = \frac{3}{4}u^4 + \frac{1}{2}u^2v + v.$



The figure of $W(l(f,\mu))$ for
$g(u,v) = (u, v^3+uv^2)$
$\mu(u,v) = \frac{1}{2}v^2+u$.



The figure of $W(l(f,\mu))$ for
$g(u,v) = (u \cdot v^3 + uv)$
$\mu(u,v) = v$.



II $n = 2$, $r = 3$.

1) B-equivalence

	$g\|\mu^{-1}(t)$	type
1)	$(u+t, v, u^2 \pm v^2)$	A_1
2)	$(3u^2+t, v, 2u^3 \pm v^2)$	A_2
3)	$(4u^3-4uv+t, v, -3u^4+v(2u^2+v))$	A_3
4)	$(5u^4-4uv+t(3u^2+1), v, 4u^5+2u^3t-v(2u^2+v))$	A_4
5)	$(3u^2+t(v+1), 3v^2+tu, 2u^3+2v^3+tuv)$	D_4^+
6)	$(3u^2-v^2+t(2u+1), -2uv+2tv, -2u^3+2uv^2-t(u^2+v^2))$	D_4^-

2) D-equivalence

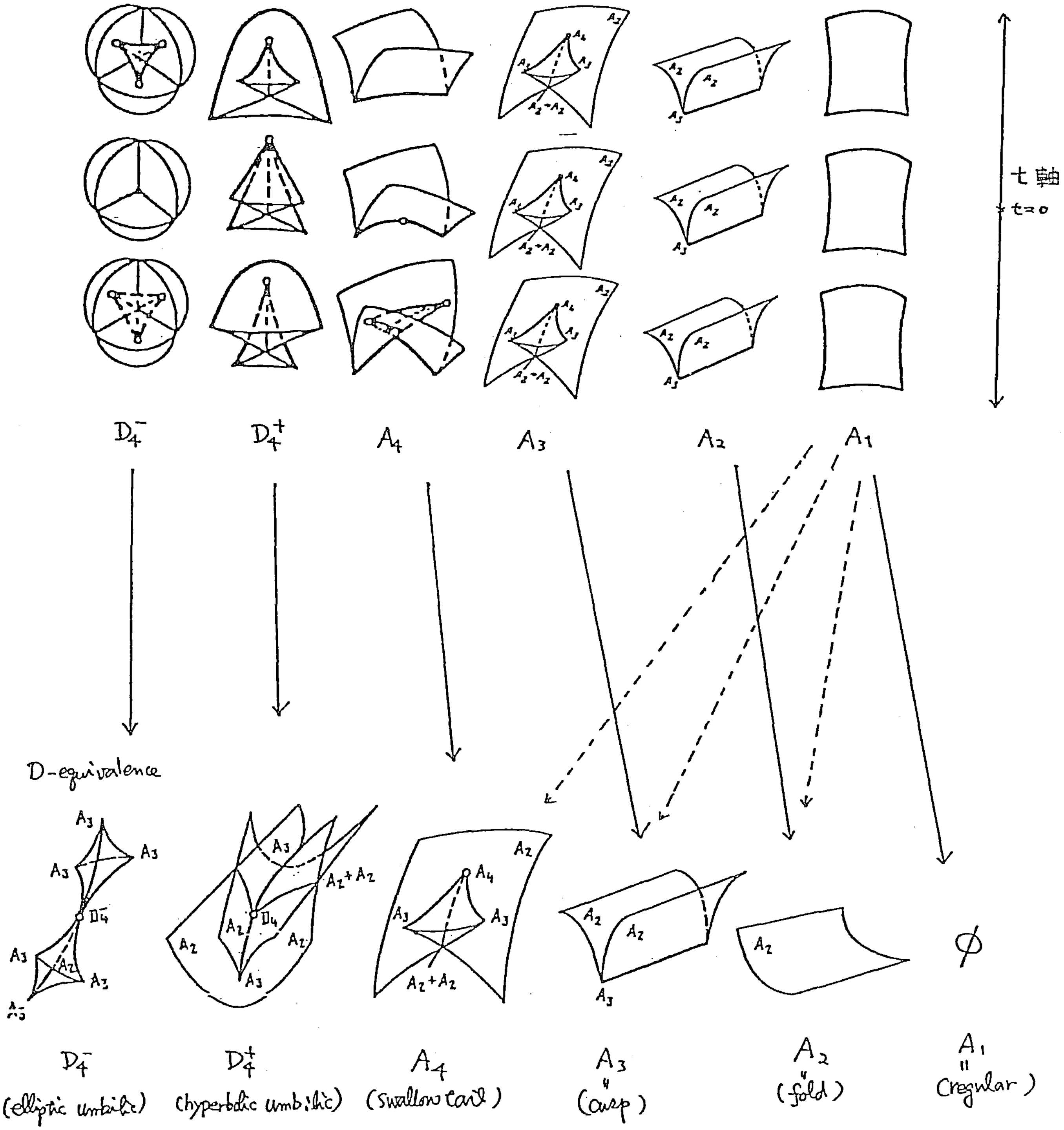
	g	μ	contact sing?	type
1)	(u, v, w)	w	reg	A_1
2)	(u^2, v, w)	$w-(1/3)u^3+v$	reg	A_2
3)	(u, v, w^2)	$w-(1/2)u+v$	sing	A_1
4)	(u^3+vu, v, w)	$(3/4)u^4+(1/2)u^2v+w$	reg	A_3
5)	(u, v, w^3+uw)	w	sing	A_1
6)	(u, v, w^3+w^2v)	$(1/2)w^2+v$	sing	A_2
7)	(u^4+u^2w+uv, v, w)	$(4/5)u^5+(1/2)u^2(v+w)+w$	reg	A_4
8)	(u^2+vw, v^2+uw, w)	$u^3+v^3+uvw+w$	reg	D_4^+
9)	$((3/4)u^2-v^2+uw, uv+vw, w)$	$u^3-uv^2+(u^2+v^2)w+w$	reg	D_4^-
10)	$(3u^2+vw, v, \pm w^2-2u^3+vw)$	w	sing	A_2
11)	(u, v, w^4+vw^2+uw)	w	sing	A_1
12)	$(u, v, w^4+w^3v+w^2u)$	$(4/3)w^3+(2/3)w^2v+u(2w+1)$	sing	A_3

3) equivalence

?

$n = 2,\ r = 3.$

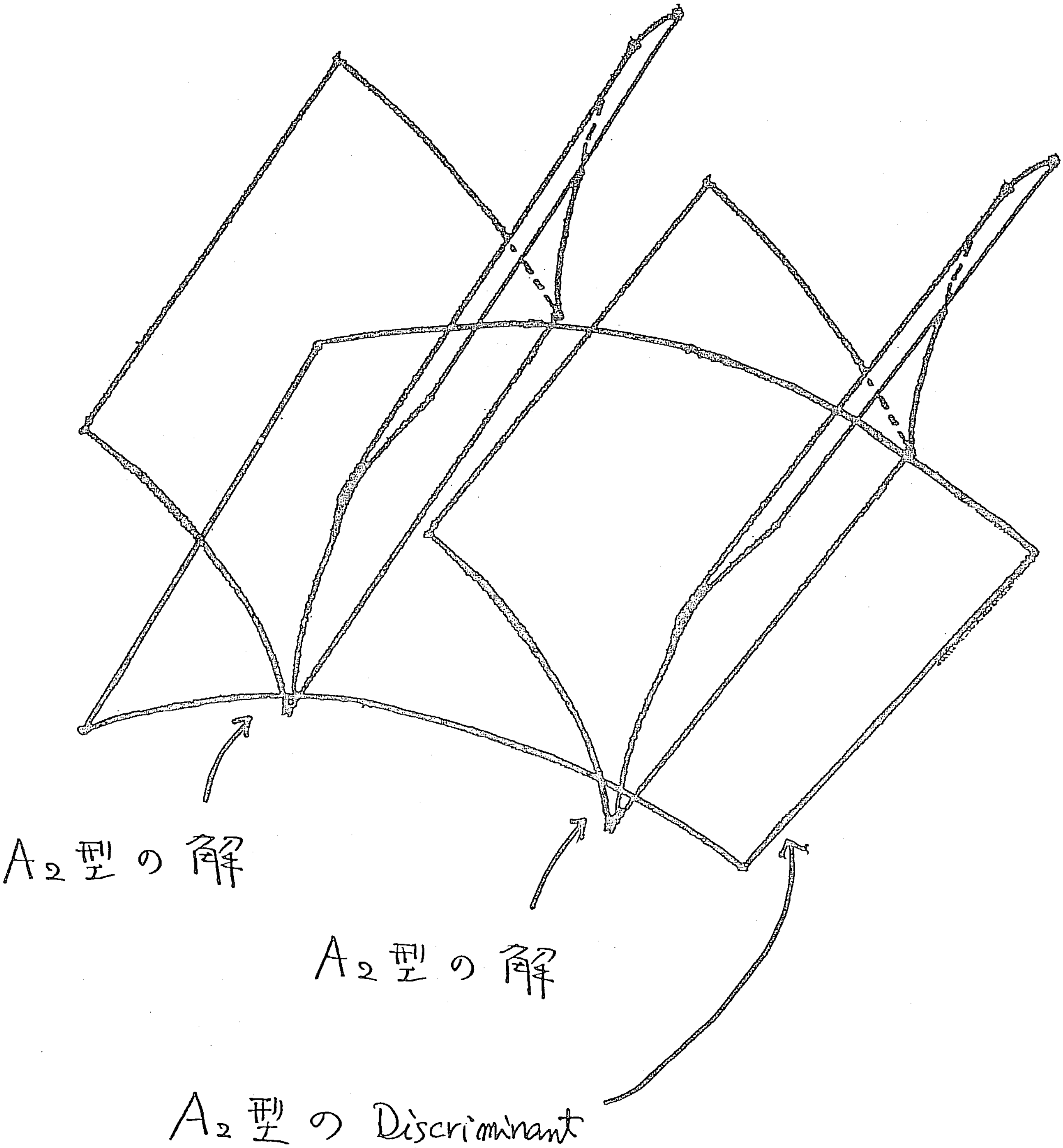
B-equivalence



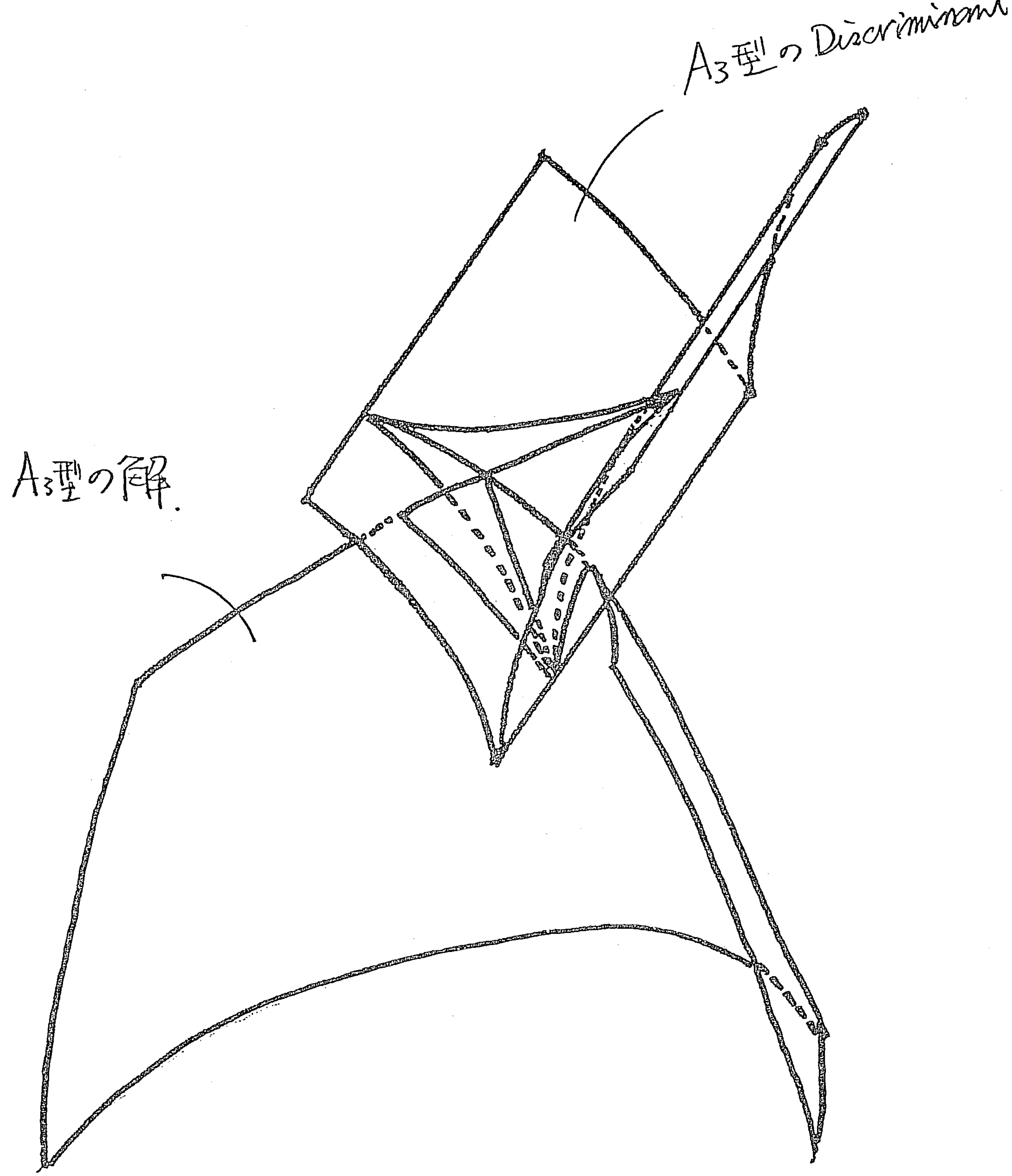
+ 6 types. (Mixed types).

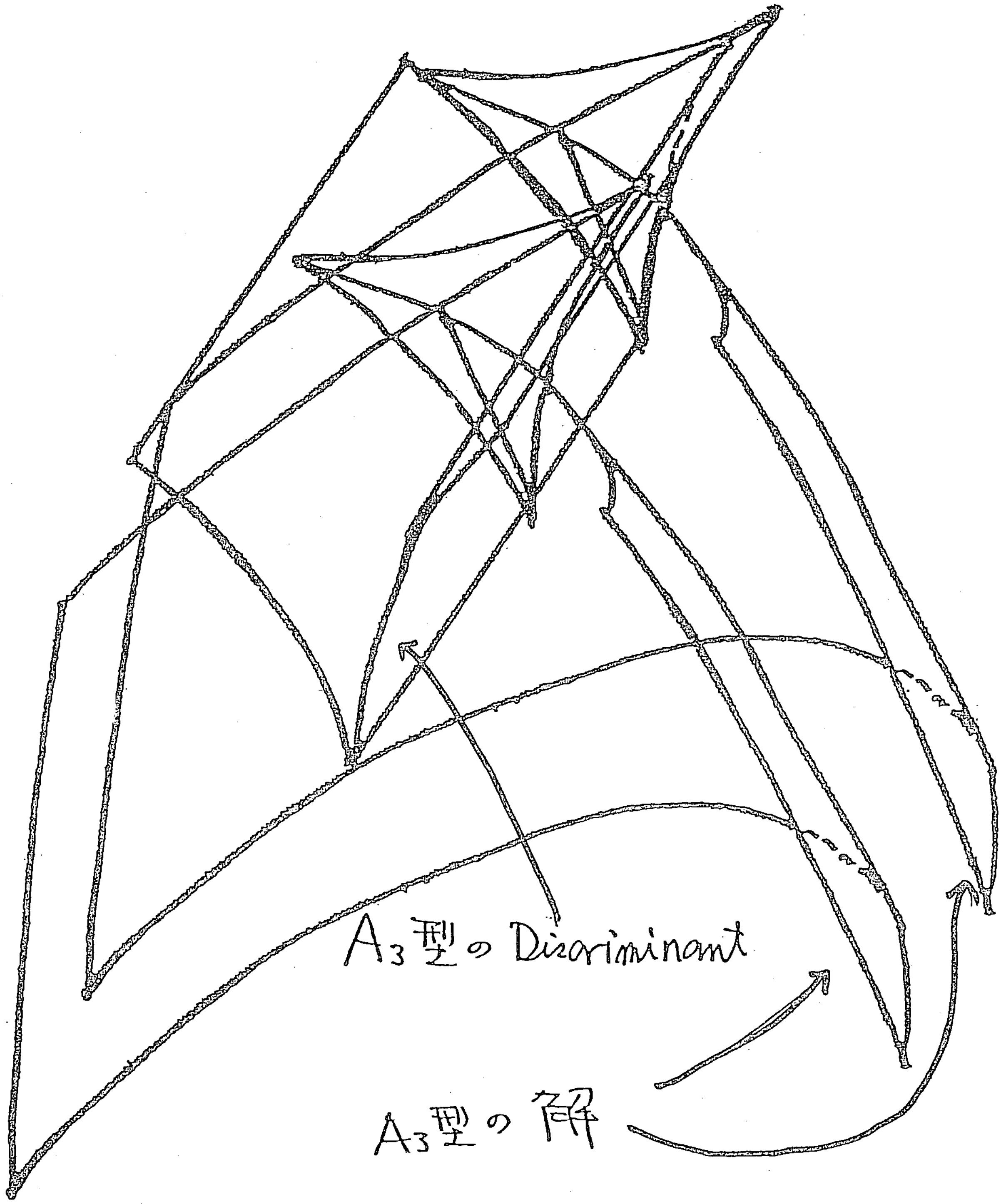
⟶ : correspondence between solutions and discriminants of contact regular equations.

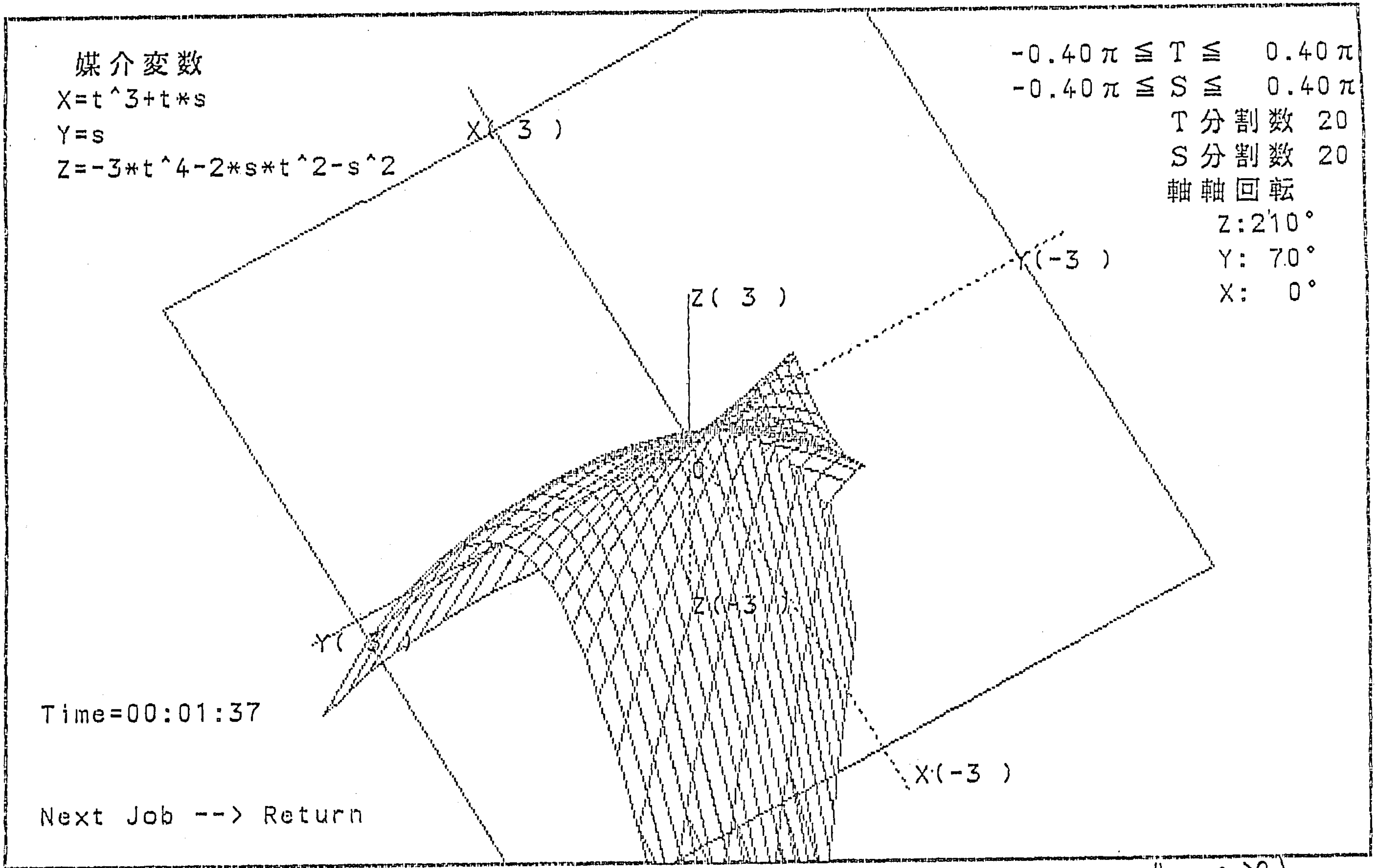
- - - -→ : correspondence between solutions and discriminants of clairaut type equations



A₂型の解
A₂型の解
A₂型の Discriminant

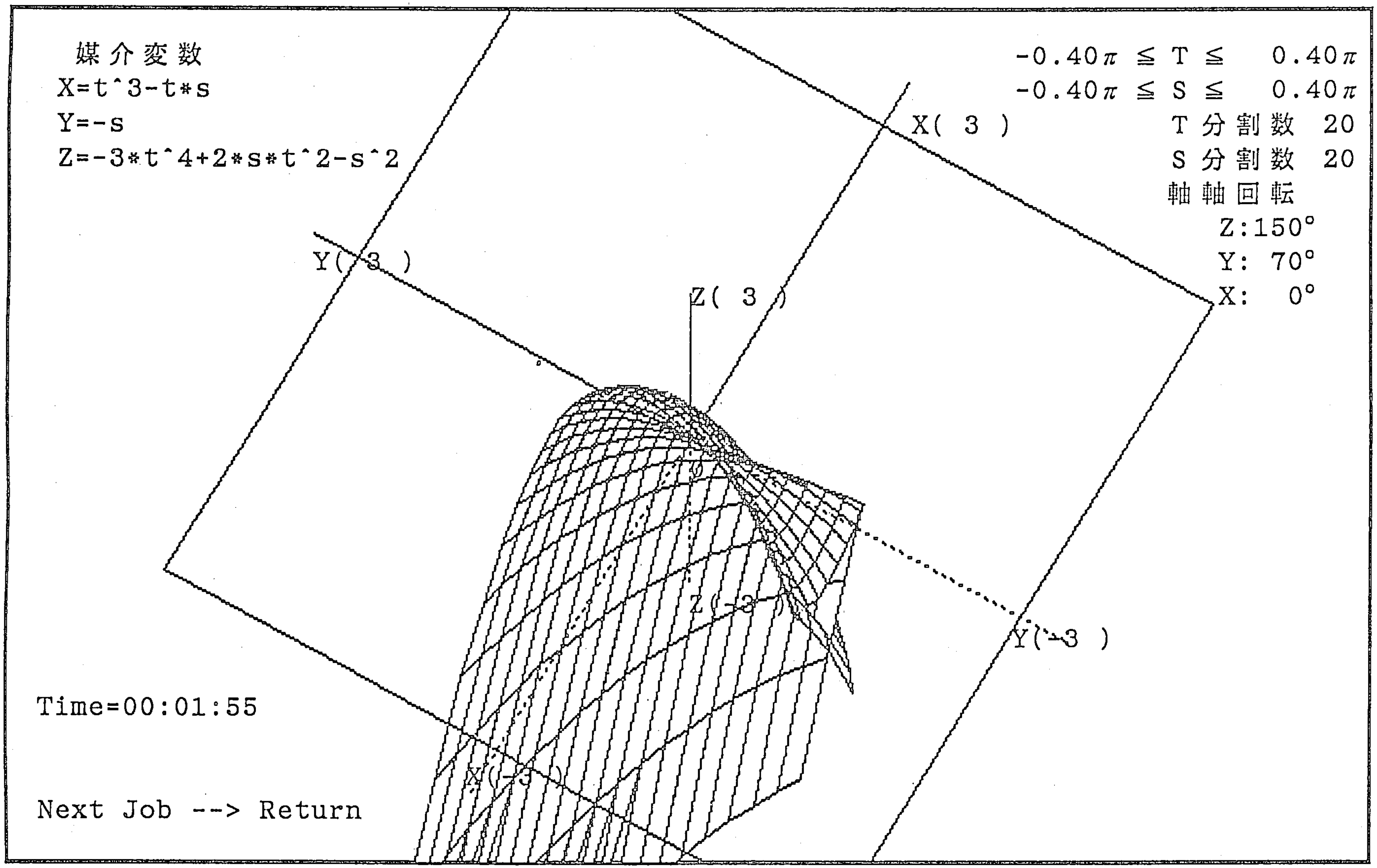

A3型の Discriminant
A3型の解.


A3型の Discriminant
A3型の解



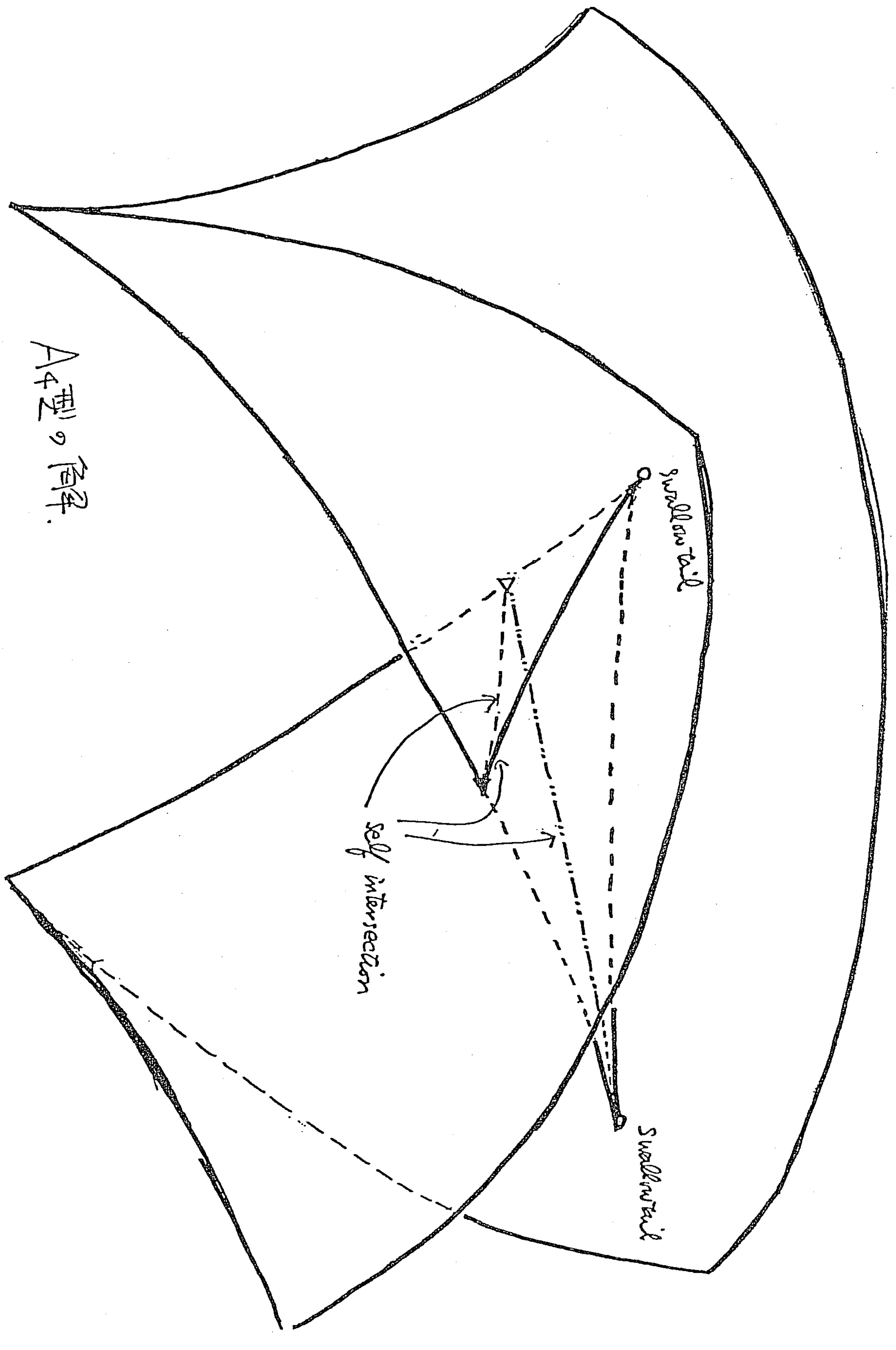
(3) $S_1^4 \pm (S_2 - S_1^2)^2 + tS_1 - x_1 S_1 - x_2 S_2 - z$ (swallow tail)

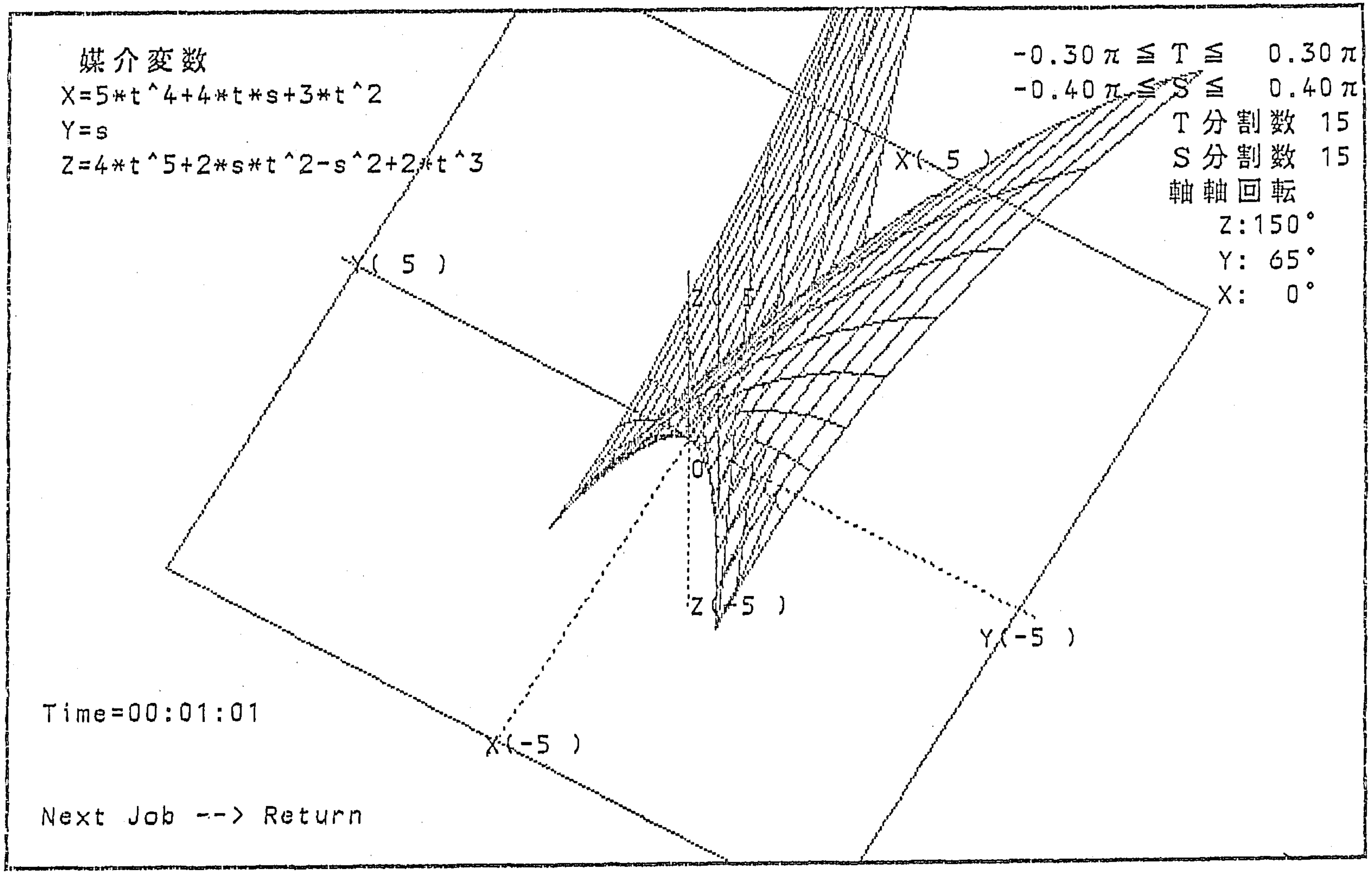
$(t, u, v) \longmapsto (u^3 \mp u \cdot v + \frac{t}{4},\ v,\ -3u^4 \pm 2v \cdot u^2 \mp v^2)$ $t=0$



(3)

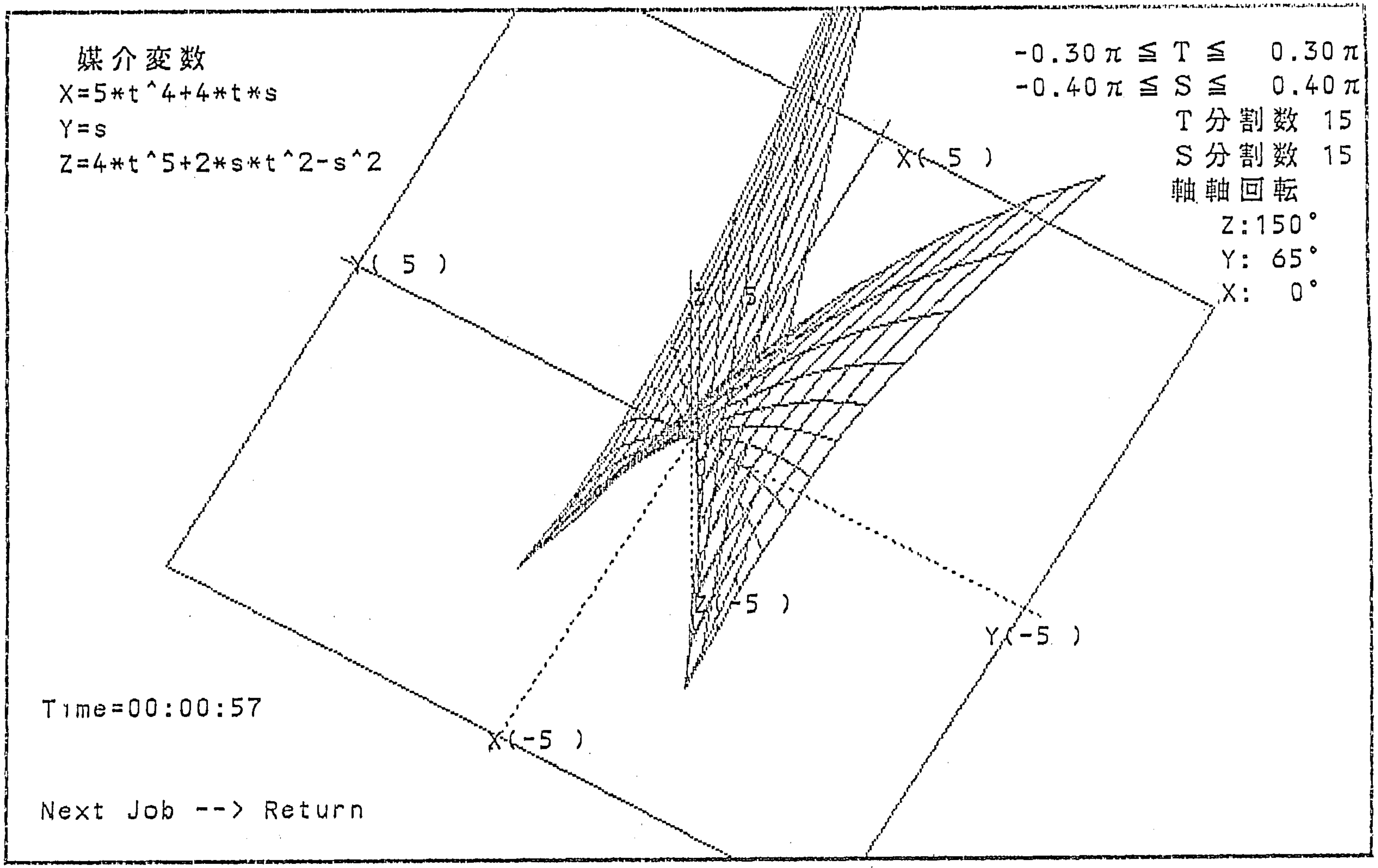
Swallow tail


A4型の解.
swallow tail
self intersection
swallowtail

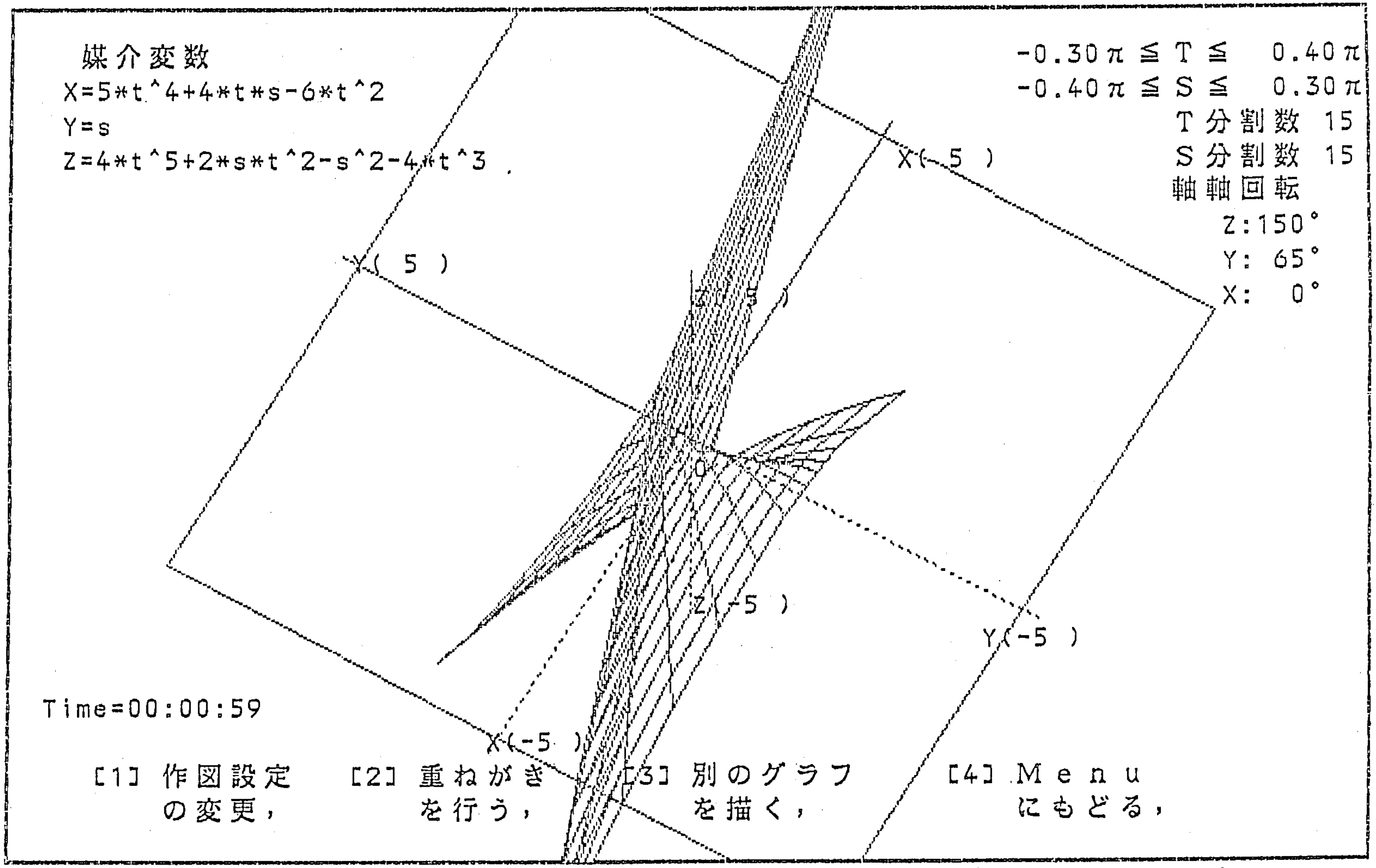

媒介変数
X=5*t^4+4*t*s+3*t^2
Y=s
Z=4*t^5+2*s*t^2-s^2+2*t^3
-0.30π ≦ T ≦ 0.30π
-0.40π ≦ S ≦ 0.40π
T 分割数 15
S 分割数 15
軸軸回転
Z:150°
Y: 65°
X: 0°
Y(5)
X(5)
Z(5)
0
Z(-5)
Y(-5)
X(-5)
Time=00:01:01
Next Job --> Return

(4)

t>0

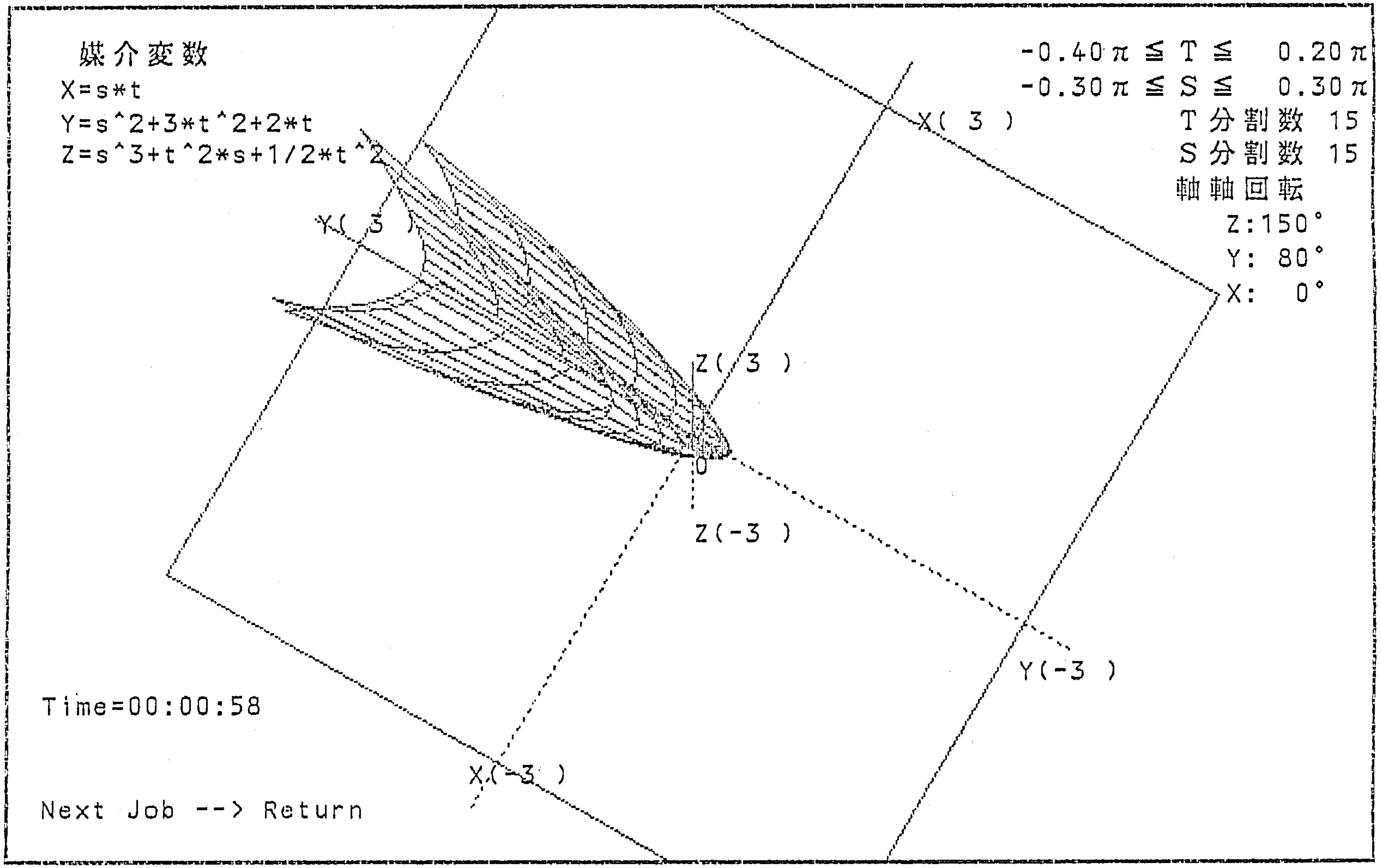

媒介変数
X=5*t^4+4*t*s
Y=s
Z=4*t^5+2*s*t^2-s^2
-0.30π ≦ T ≦ 0.30π
-0.40π ≦ S ≦ 0.40π
T 分割数 15
S 分割数 15
軸軸回転
Z:150°
Y: 65°
X: 0°
X(5)
Y(5)
Z(5)
Z(-5)
Y(-5)
X(-5)
Time=00:00:57
Next Job --> Return

(4)　　t=0

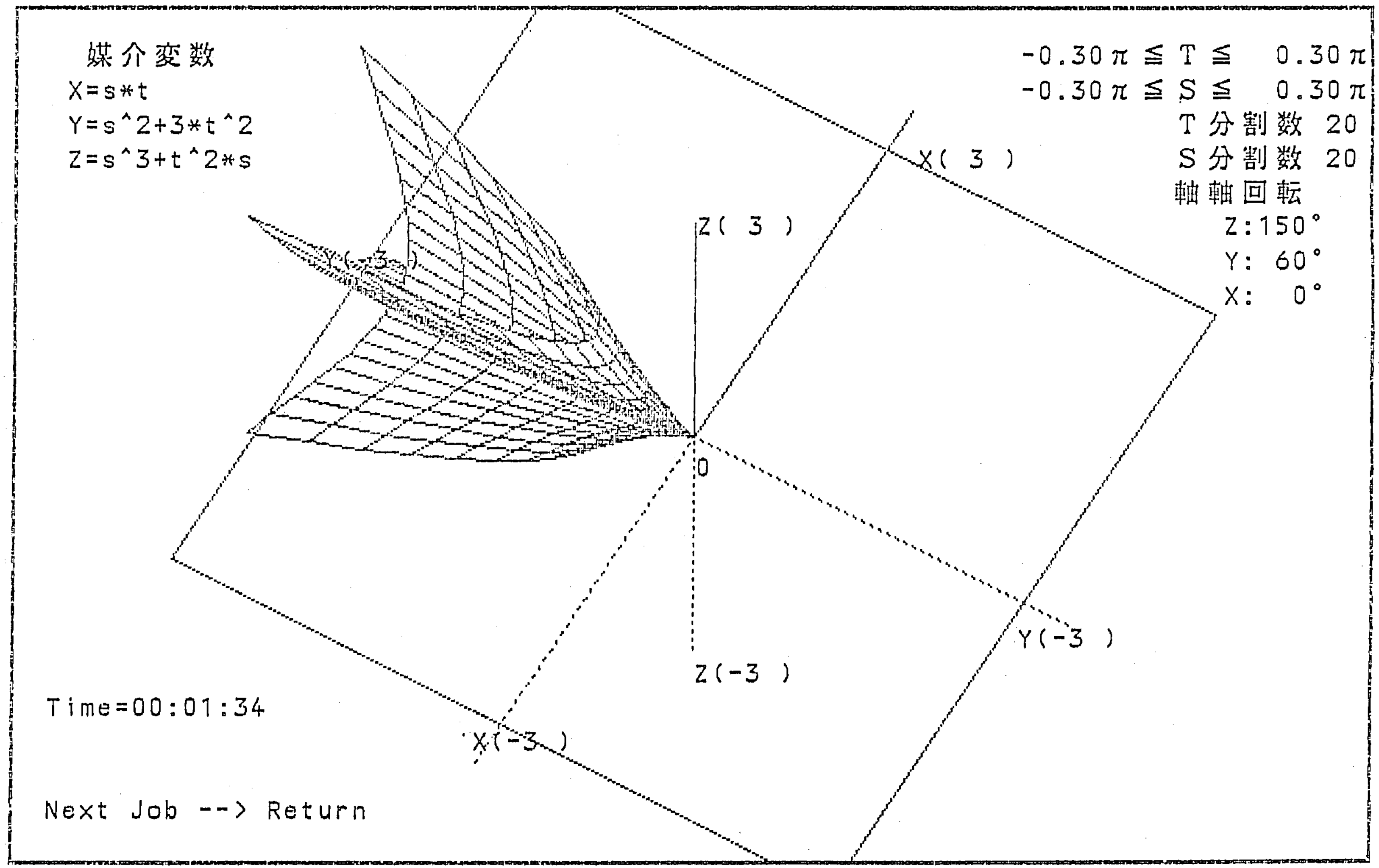

媒介変数
X=5*t^4+4*t*s-6*t^2
Y=s
Z=4*t^5+2*s*t^2-s^2-4*t^3
-0.30π ≦ T ≦ 0.40π
-0.40π ≦ S ≦ 0.30π
T分割数 15
S分割数 15
軸軸回転
Z:150°
Y: 65°
X: 0°
Y(5)
X(-5)
Z(5)
0
Z(-5)
Y(-5)
X(-5)
Time=00:00:59
[1] 作図設定の変更，
[2] 重ねがきを行う，
[3] 別のグラフを描く，
[4] Menuにもどる，

(4)

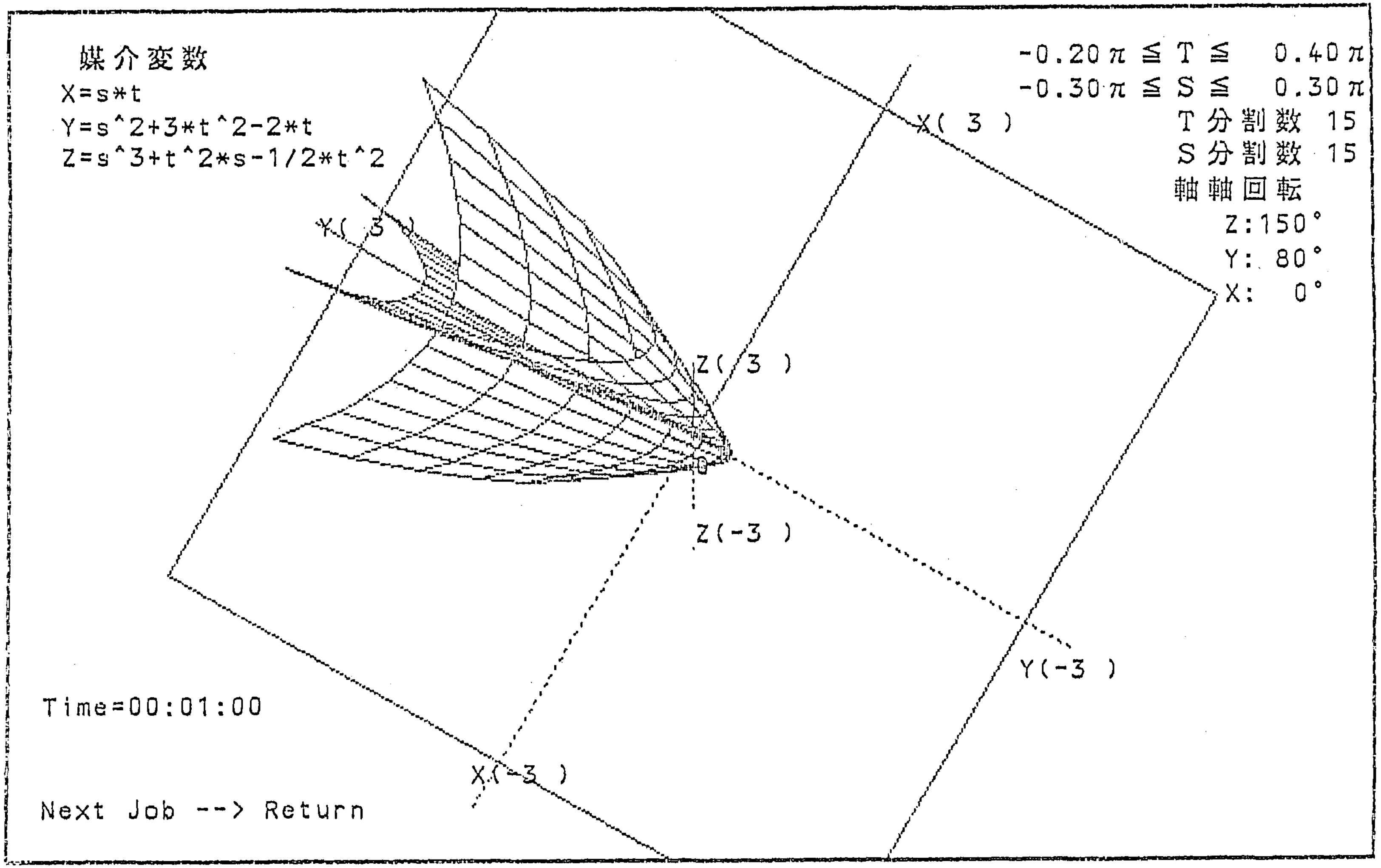
t<0


媒介変数
X=s*t
Y=s^2+3*t^2+2*t
Z=s^3+t^2*s+1/2*t^2
-0.40π ≦ T ≦ 0.20π
-0.30π ≦ S ≦ 0.30π
T分割数 15
S分割数 15
軸軸回転
Z:150°
Y: 80°
X: 0°
X(3)
Y(3)
Z(3)
0
Z(-3)
Y(-3)
X(-3)
Time=00:00:58
Next Job --> Return

5)


媒介変数
X=s*t
Y=s^2+3*t^2
Z=s^3+t^2*s
-0.30π ≦ T ≦ 0.30π
-0.30π ≦ S ≦ 0.30π
T分割数 20
S分割数 20
軸軸回転
Z:150°
Y: 60°
X: 0°
X(3)
Z(3)
Y(-3)
0
Y(-3)
Z(-3)
X(-3)
Time=00:01:34
Next Job --> Return

5)


媒介変数
X=s*t
Y=s^2+3*t^2-2*t
Z=s^3+t^2*s-1/2*t^2
-0.20π ≦ T ≦ 0.40π
-0.30π ≦ S ≦ 0.30π
T分割数 15
S分割数 15
軸軸回転
Z:150°
Y: 80°
X: 0°
X(3)
Y(3)
Z(3)
Z(-3)
Y(-3)
X(-3)
Time=00:01:00
Next Job --> Return

5)

III $n = 2$, $r = 4$.

1) B-equivalence ($t = (t_1, t_2)$)

	$g\|\mu^{-1}((t_1,t_2))$
1)	$(u+t_1, v+t_2, u^2 \pm v^2)$
2)	$(3u^2+t_1, v+t_2, 2u^3 \pm v^2)$
3)	$(4u^3-4uv+t_1, 2v+t_2, 3u^4-v(2u^2+v))$
4)	$(5u^4-4uv+t_1(3u^2+1), 2v+t_2, 4u^5+2u^3t_1-v(2u^2+v))$
5)	$(u^2+(1/3)t_1(v+1), v^2+(1/3)(t_1u+t_2), u^3+v^3+(1/2)t_1uv)$
6)	$(3u^2-v^2+t_1(v+1), uv-t_1v-(1/2)t_2, u^3-uv^2+(1/2)t_1(u^2+v^2))$
7)	$(6u^5-4uv+t_1(3u^2+1)+4t_2u^3, 2v+t_2, -5u^6+v(2u^2-v)-2t_1u^3-3t_2u^4)$
8)	$(2uv+t_1(2u+1), u^2+4v^3+t_2(2v+1), 2u^2v+3v^4+t_1u^2+t_2v^2)$

2) D-equivalence

?

3) equivalence

?

References

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＊ グラフィックスについては，自作のプログラムの他に
以下のプログラムを使いました．
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